Translating Strategic Information Across Selection Games

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Three Topological Properties

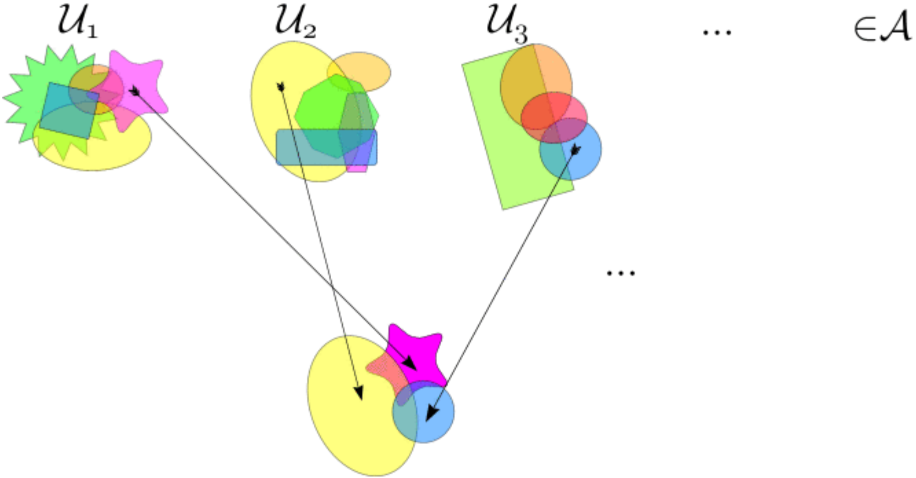
Suppose X is a topological space.

- **Lindelöf**: every open cover of X has a countable subcover.
- **Separable**: X has a countable dense subset.
- **Fréchet-Urysohn (Arhangel'skii 1963)**: if x is in the closure of A, then there is a countable $B \subseteq A$ so that x is in the closure of B.

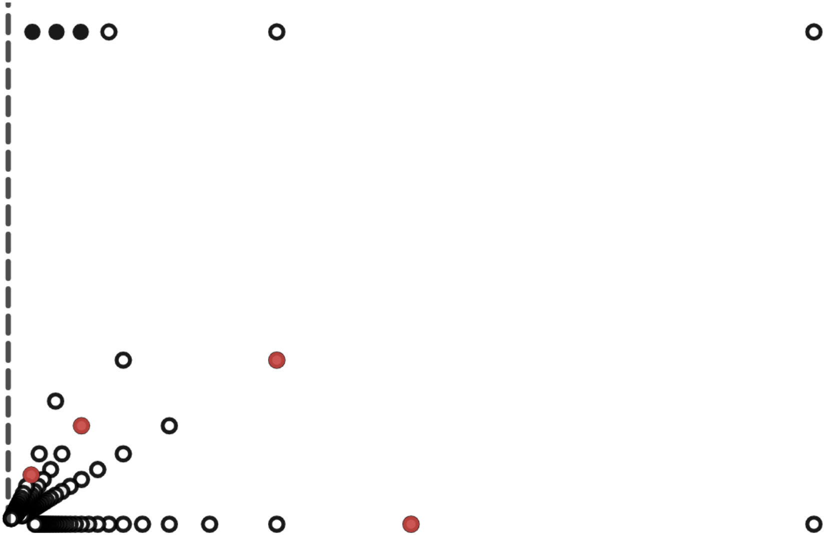
Selection Principles

- **Rothberger (1938)**: Given a sequence of open covers $\mathscr{U}_n$, sets $U_n \in \mathscr{U}_n$ can be chosen so that $\{U_n : n \in \omega\}$ is an open cover.
- **Selective Separability (Scheepers 1999)**: Given a sequence of dense sets, D_n, points $x_n \in D_n$ can be chosen so that $\{x_n : n \in \omega\}$ is dense.
- **Strong Countable Fan Tightness (Arhangel'skii 1986)**: Given a point x and a sequence of sets A_n so that $x \in \overline{A_n}$, points $x_n \in A_n$ can be chosen so that $x \in \overline{\{x_n : n \in \omega\}}$.

Rothberger



Strong Countable Fan Tightness



Selection Principles

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are collections.

$S_1(\mathcal{A},\mathcal{B})$ (Menger, Hurewicz 1924) (Scheepers 1996)

$S_1(\mathcal{A},\mathcal{B})$ means that for all sequences A_n consisting of elements of $\mathcal{A}$, there are choices $x_n \in A_n$ so that $\{x_n : n \in \omega\} \in \mathcal{B}$.

Notation

- Let $\mathcal{O}(X)$ denote the open covers of X.
- Let $\mathscr{D}_X$ denote the dense subsets of X.
- Let $\Omega_{X,x}$ denote the sets $A \subseteq X$ such that $x \in \overline{A}$.

Note that

- X is Rothberger if and only if $S_1(\mathcal{O}(X), \mathcal{O}(X))$.
- X is selectively separable if and only if $S_1(\mathscr{D}_X, \mathscr{D}_X)$.
- X has strong countable fan tightness at x if and only if $S_1(\Omega_{X,x}, \Omega_{X,x})$.

Selection Games

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are collections of sets.

$G_1(\mathcal{A}, \mathcal{B})$

At round n, player I plays $A_n \in \mathcal{A}$ and player II plays $x_n \in \mathcal{A}_n$.

I	$A_0 \in \mathcal{A}$	$A_1 \in \mathcal{A}$	$\cdots$
II	$x_0 \in A_0$	$x_1 \in A_1$	$\cdots$

- Player II wins a given run of the game if $\{x_n : n \in \omega\} \in \mathcal{B}$.
- Player I wins a given run of the game if $\{x_n : n \in \omega\} \notin \mathcal{B}$.

Strategies

- A **perfect information (PI) strategy** for either player One or Two is a strategy for responding to the other player that takes as inputs all of the previous plays of the game.
- A **Markov strategy** for player Two is a strategy that takes as inputs the current turn number and the most recent play of player One.
- A **pre-determined (PD) strategy** for player One is a strategy where the only input is the current turn number.
- A strategy is **winning** if following the strategy guarantees that the player will win the game.

Strategies

- Playing according to a PI strategy for One:

I	$\sigma(\emptyset)$	$\sigma(x_0)$	$\sigma(x_0, x_1)$	$\cdots$
II	x_0	x_1	x_2	$\cdots$

- Playing according to a PI strategy for Two:

I	A_0	A_1	A_2	$\cdots$
II	$\tau(A_0)$	$\tau(A_0, A_1)$	$\tau(A_0, A_1, A_2)$	$\cdots$

- Playing according to a Markov strategy for Two:

I	A_0	A_1	A_2	$\cdots$
II	$\tau(A_0, 0)$	$\tau(A_1, 1)$	$\tau(A_2, 2)$	$\cdots$

- Playing according to a PD strategy for One:

I	$\sigma(0)$	$\sigma(1)$	$\sigma(2)$	$\cdots$
II	x_0	x_1	x_2	$\cdots$

A Strength Hierarchy

For the selection game $G_1(\mathcal{A}, \mathcal{B})$:

$$\begin{array}{rcl} \text{Two has a winning Markov Strategy} & & \\ \Downarrow & & \\ \text{Two has a winning PI strategy} & & \\ \Downarrow & & \\ \text{One has no winning PI strategy} & & \\ \Downarrow & & \\ \text{One has no winning PD strategy} & \iff & S_1(\mathcal{A}, \mathcal{B}) \end{array}$$

A Strength Hierarchy

For the Rothberger game:

$$\begin{array}{rcl} \text{Two has a winning Markov Strategy} & \iff & \text{countable} \\ \Downarrow & & \\ \text{Two has a winning PI strategy} & & \\ \Downarrow & & \\ \text{One has no winning PI strategy} & & \\ \Downarrow & & \\ \text{One has no winning PD strategy} & \iff & \text{Rothberger} \end{array}$$

Game Equivalence

Definition

Define $G_1(\mathcal{A}, \mathcal{C}) \leq_{\mathrm{II}} G_1(\mathcal{B}, \mathcal{D})$ as the conjunction of the following implications.

1. Two has Mark in $G_1(\mathcal{A}, \mathcal{C}) \implies$ Two has a Mark in $G_1(\mathcal{B}, \mathcal{D})$
2. Two has PI strat in $G_1(\mathcal{A}, \mathcal{C}) \implies$ Two has a PI strat in $G_1(\mathcal{B}, \mathcal{D})$
3. One has no PI strat in $G_1(\mathcal{A}, \mathcal{C}) \implies$ One has no PI strat in $G_1(\mathcal{B}, \mathcal{D})$
4. One has no PD strat in $G_1(\mathcal{A}, \mathcal{C}) \implies$ One has no PD strat in $G_1(\mathcal{B}, \mathcal{D})$

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3. One has no PI strat in $G_1(\mathcal{A}, \mathcal{C}) \implies$ One has no PI strat in $G_1(\mathcal{B}, \mathcal{D})$
4. One has no PD strat in $G_1(\mathcal{A}, \mathcal{C}) \implies$ One has no PD strat in $G_1(\mathcal{B}, \mathcal{D})$

- This relation is transitive.
- if $G_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_1(\mathcal{B}, \mathcal{D})$ and $G_1(\mathcal{B}, \mathcal{D}) \leq_{\text{II}} G_1(\mathcal{A}, \mathcal{C})$, we say the two games are **equivalent**.

Roots in Earlier Work

Monotonicity Laws (Scheepers 2003)

Suppose $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are collections.

- If $\mathcal{A} \subseteq \mathcal{B}$, then $G_1(\mathcal{B}, \mathcal{C}) \leq_{\text{II}} G_1(\mathcal{A}, \mathcal{C})$.
- If $\mathcal{C} \subseteq \mathcal{D}$, then $G_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_1(\mathcal{A}, \mathcal{D})$.

So

- $G_1(\mathcal{O}(X), \mathcal{O}(X)) \leq_{\text{II}} G_1(\text{countable open covers}, \mathcal{O}(X))$ and
- $G_1(\mathscr{D}_X, \mathscr{D}_X) \leq_{\text{II}} G_1(\mathscr{D}_X, \Omega_{X,x})$.

The Space of Continuous Functions

Set $C(X)$ to be collection of all continuous functions $f : X \to \mathbb{R}$.

The Topology of Point-Wise Convergence

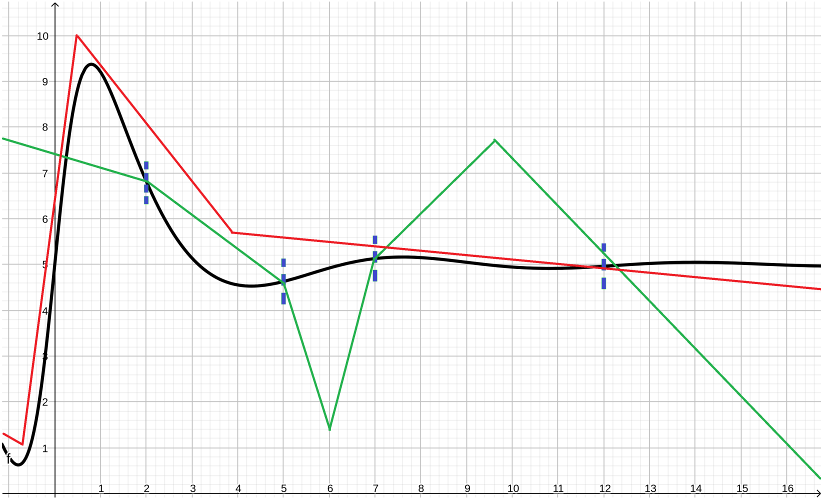
$C(X)$ with this topology will be denoted $C_p(X)$. The open sets are generated by sets of the form:

$$[f; \{x_0, \cdots, x_n\}, \varepsilon] = \{g : |f(x_0) - g(x_0)| < \varepsilon, \cdots, |f(x_n) - g(x_n)| < \varepsilon\}$$

where f is continuous, $x_0, \cdots, x_n \in X$, and $\varepsilon > 0$.

Common Topologies on the Space of Continuous Functions

f with a neighborhood $[f; F, \varepsilon]$.



A Topological Connection

- If $f : X \to \mathbb{R}$ is continuous, and $k \in \omega$, then $f^{-1}[(-2^{-k}, 2^{-k})]$ is open.
- If $A \subseteq C_p(X)$ has $\mathbf{0}$ in its closure, then for a fixed k,

$$\mathscr{U} = \{f^{-1}[(-2^{-k}, 2^{-k})] : f \in A\}$$

  is an open cover of X.
- In fact, if $F \subseteq X$ is finite, then there is a $U \in \mathscr{U}$ so that $F \subseteq U$.

ω-Covers

A non-trivial open cover $\mathscr{U}$ of X is an ω**-cover** if for all finite $F \subseteq X$, there is a $U \in \mathscr{U}$ so that $F \subseteq U$.
$\Omega(X)$ is the collection of ω-covers of X.

Proposition

If $A \in \Omega_{C_p(X),\mathbf{0}}$, then

$$\mathscr{U}(A,k) := \{f^{-1}[(-2^{-k}, 2^{-k})] : f \in A\} \in \Omega(X)$$

Proof.

Suppose $A \in \Omega_{C_p(X),\mathbf{0}}$. Let $F \subseteq X$ be finite. Then $[\mathbf{0}; F, 2^{-k}]$ is an open nhood of $\mathbf{0}$. So there is an $f \in A$ so that $f \in [\mathbf{0}; F, 2^{-k}]$. Then $F \subseteq f^{-1}[(-2^{-k}, 2^{-k})]$.

□

ω-Covers

Proposition

Consider $f_n \in C_p(X)$ with the property that

$$\{f_n^{-1}[(-2^{-n}, 2^{-n})] : n \in \omega\} \in \Omega(X).$$

Then we also know that $\{f_n : n \in \omega\} \in \Omega_{C_p(X),\mathbf{0}}$.

Proof.

Consider a basic open nhood $[\mathbf{0}; F, \varepsilon]$. Since F is finite, there is an n so that $2^{-n} < \varepsilon$ and $F \subseteq f_n^{-1}[(-2^{-n}, 2^{-n})]$. Thus $f_n \in [\mathbf{0}; F, \varepsilon]$. □

Complete Regularity

Definition

Recall that space X is **completely regular** if for every point $x \in X$ and closed set $F \subseteq X$ with $x \notin F$, there is a continuous function $f : X \to [0,1]$ so that $f(x) = 0$ and $f|_F = \mathbf{1}$.

Note that you can also find continuous functions to separate finite sets from closed sets, and even separate compact sets from closed sets.

Complete Regularity

Suppose that X is Hausdorff and completely regular.

Proposition

If $\mathscr{U} \in \Omega(X)$, then

$$A(\mathscr{U}) := \{f : (\exists U \in \mathscr{U})[f|_{X \setminus U} = \mathbf{1}]\} \in \Omega_{C_p(X),\mathbf{0}}.$$

Proof.

Suppose $\mathscr{U} \in \Omega(X)$. Consider a basic open nhood $[\mathbf{0}; F, \varepsilon]$. There is a $U \in \mathscr{U}$ so that $F \subseteq U$. There is then a continuous $f : X \to \mathbb{R}$ so that $f|_F = \mathbf{0}$ and $f|_{X \setminus U} = \mathbf{1}$. Then $f \in [\mathbf{0}; F, \varepsilon]$. □

Complete Regularity

Proposition

Suppose $\{f_n : n \in \omega\} \in \Omega_{C_p(X),\mathbf{0}}$ and that $U_n \subseteq X$ are open so that $f_n|_{X \setminus U_n} = \mathbf{1}$. Then $\{U_n : n \in \omega\} \in \Omega(X)$.

Proof.

Suppose $F \subseteq X$ is finite. Then $[\mathbf{0}; F, 1]$ is open nhood of $\mathbf{0}$, so there is an n so that $f_n \in [\mathbf{0}; F, 1]$. Thus $F \subseteq f_n^{-1}[(-1,1)]$. $f_n|_{X \setminus U_n} = \mathbf{1}$, this means that $F \cap (X \setminus U_n) = \emptyset$. Therefore $F \subseteq U_n$. □

Equivalences

Suppose that X is Hausdorff and completely regular.

- X is countable if and only if $C_p(X)$ is Frechét-Urysohn (at $\mathbf{0}$).
- $S_1(\Omega(X), \Omega(X))$ if and only if $S_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$ (Arkhangel'skii 1978).

Theorem (Clontz and Holshouser 2018)

$G_1(\Omega(X), \Omega(X))$ and $G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$ are equivalent.

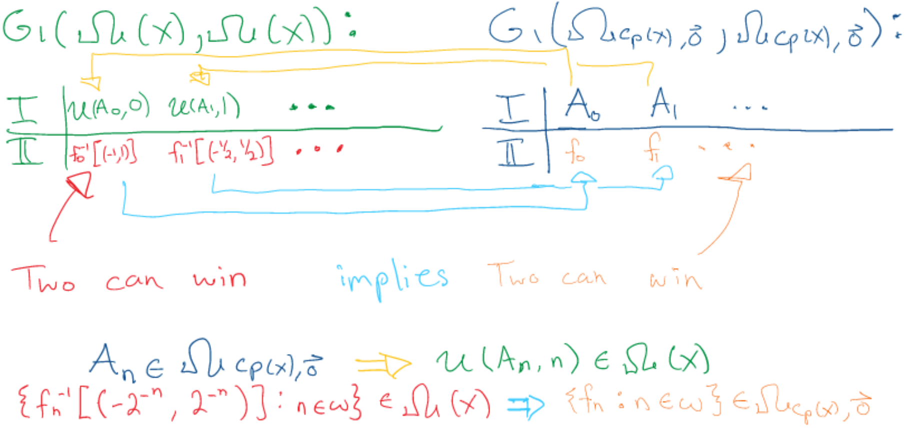
A Repetitive Proof

To see that if Two has a winning PI strat in $G_1(\Omega(X), \Omega(X))$, then Two also has one in $G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$:

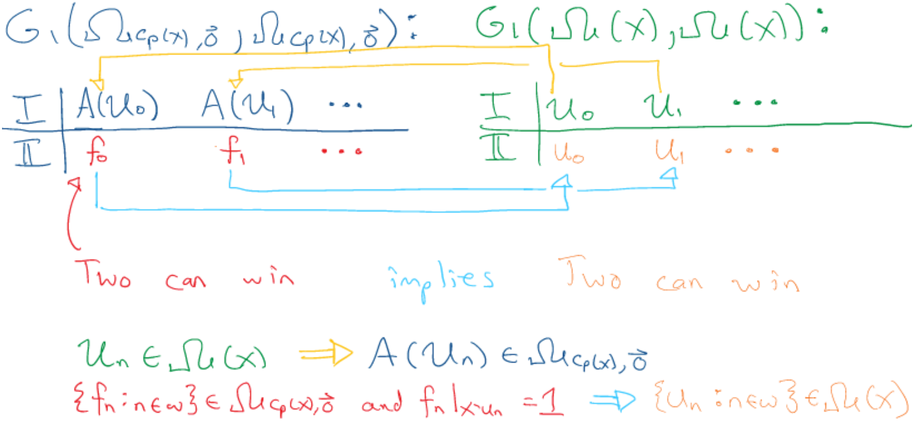
1. Let τ win for Two in $G_1(\Omega(X), \Omega(X))$ and use it to define t for Two in $G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$.
2. If One plays A_0, this creates $\mathscr{U}(A_0, 0)$, which builds $\tau(\mathscr{U}(A_0, 0))$.
3. $\tau(\mathscr{U}(A_0, 0))$ is $f_0^{-1}[(-1, 1)]$ for some $f_0 \in A_0$. Set $t(A_0) = f_0$.
4. Now suppose One responds with A_1. This creates $\mathscr{U}(A_1, 1)$, which builds $\tau(\mathscr{U}(A_0, 0), \mathscr{U}(A_1, 1))$, which comes from $f_1 \in A_1$. Set $t(A_0, A_1) = f_1$.
5. Continue in this way recursively to define all of t.
6. Verify t is well-defined.
7. Verify t is a winning strategy.

Now do essentially the same thing with the other three parts.

$G_1(\Omega(X), \Omega(X)) \leq_{\mathrm{II}} G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$



$G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}}) \leq_{\mathrm{II}} G_1(\Omega(X), \Omega(X))$



A Translation Theorem

Theorem (Caruvana and Holshouser 2019)

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be collections. Suppose there are functions

- $\overleftarrow{T}_{\mathrm{I},n} : \mathcal{B} \to \mathcal{A}$ and
- $\overrightarrow{T}_{\mathrm{II},n} : \bigcup \mathcal{A} \times \mathcal{B} \to \bigcup \mathcal{B}$

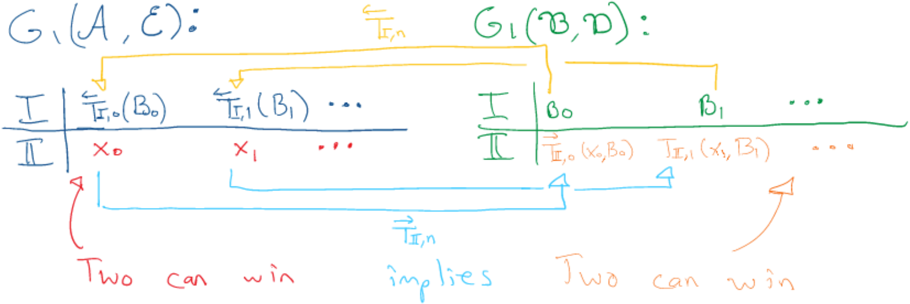
so that

1. if $x \in \overleftarrow{T}_{\mathrm{I},n}(B)$, then $\overrightarrow{T}_{\mathrm{II},n}(x, B) \in B$ and
2. if $x_n \in \overleftarrow{T}_{\mathrm{I},n}(B_n)$ for all n, then

$$\{x_n : n \in \omega\} \in \mathcal{C} \implies \{\overrightarrow{T}_{\mathrm{II},n}(x_n, B_n) : n \in \omega\} \in \mathcal{D}$$

Then $G_1(\mathcal{A}, \mathcal{C}) \leq_{\mathrm{II}} G_1(\mathcal{B}, \mathcal{D})$.

A Translation Theorem



① $x \in \overleftarrow{T}_{I,n}(\mathcal{B}) \Rightarrow \overrightarrow{T}_{II,n}(x, \mathcal{B}) \in \mathcal{B}$

② $x_n \in \overleftarrow{T}_{I,n}(B_n)$ and $\{x_n : n \in \omega\} \in \mathcal{E} \Rightarrow \{\overrightarrow{T}_{II,n}(x_n, B_n) : n \in \omega\} \in \mathcal{D}$

Examples

$G_1(\Omega(X), \Omega(X)) \leq_{\mathrm{II}} G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$

- $\mathcal{A} = \mathcal{C} = \Omega(X)$, their unions are all open subsets of X.
- $\mathcal{B} = \mathcal{D} = \Omega_{C_p(X),\mathbf{0}}$, their unions are $C_p(X)$.
- $\overleftarrow{T}_{\mathrm{I},n}(A) = \mathscr{U}(A, n)$.
- $\overrightarrow{T}_{\mathrm{II},n}(U, A) = f$ so that $U = f^{-1}[(-2^{-n}, 2^{-n})]$.(*)

$G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}}) \leq_{\mathrm{II}} G_1(\Omega(X), \Omega(X))$

- $\mathcal{A} = \mathcal{C} = \Omega_{C_p(X),\mathbf{0}}$, their unions are $C_p(X)$.
- $\mathcal{B} = \mathcal{D} = \Omega(X)$, their unions are all open subsets of X.
- $\overleftarrow{T}_{\mathrm{I},n}(\mathscr{U}) = A(\mathscr{U})$.
- $\overrightarrow{T}_{\mathrm{II},n}(f, \mathscr{U}) = U$ so that $U \in \mathscr{U}$ and $f|_{X \setminus U} = \mathbf{1}$.(*)

(*): minus some small details.

A Synchronized Translation Theorem

Corollary (Caruvana and Holshouser 2020)

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be collections.
Suppose there is a map $\varphi : \bigcup \mathcal{B} \times \omega \to \bigcup \mathcal{A}$ so that

1. For all $B \in \mathcal{B}$ and all n, $\varphi[B \times \{n\}] \in \mathcal{A}$, and
2. If $y_n \in B_n$ for each n and $\{\varphi(y_n, n) : n \in \omega\} \in \mathcal{C}$, then $\{y_n : n \in \omega\} \in \mathcal{D}$.

Then $G_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_1(\mathcal{B}, \mathcal{D})$.

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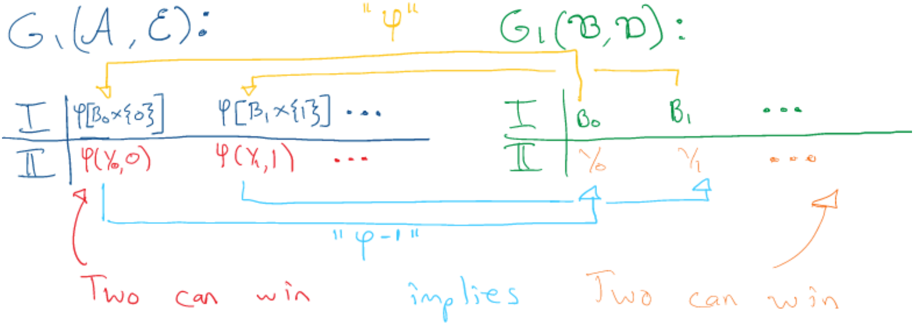
2. If $y_n \in B_n$ for each n and $\{\varphi(y_n, n) : n \in \omega\} \in \mathcal{C}$, then $\{y_n : n \in \omega\} \in \mathcal{D}$.

Then $G_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_1(\mathcal{B}, \mathcal{D})$.

To see that $G_1(\Omega(X), \Omega(X)) \leq_{\text{II}} G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$, we can use $\varphi(f, n) = f^{-1}[(-2^{-n}, 2^{-n})]$.

A Synchronized Translation Theorem



① $B \in \mathcal{B}$ and $n \in \omega$ $\Rightarrow$ $\varphi[B \times \{n\}] \in \mathcal{A}$

② $\gamma_n \in B_n$ and $\{\varphi(\gamma_n, n) : n \in \omega\} \in \mathcal{E}$ $\Rightarrow$ $\{\gamma_n : n \in \omega\} \in \mathcal{D}$

The Fell Topology on the Closed Subsets of X

Let $\mathbb{F}(X)$ denote the collection of closed subsets of X.

The Upper Fell Topology (Fell 1962)

The **Upper Fell Topology** on $\mathbb{F}(X)$ is generated by basic open sets of the form

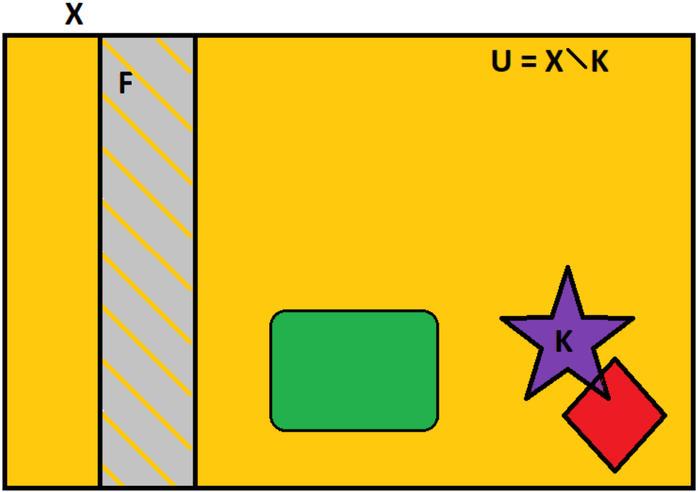
$$(U)^+ := \{F \in \mathbb{F}(X) : F \subseteq U\}$$

where U is the complement of a compact set.

A basic neighborhood of F has the form $(X \setminus K)^+$ where K is compact and $F \cap K = \emptyset$.

The Fell Topology on the Closed Subsets of X

F with an open neighborhood $(X \setminus K)^+$.



The Fell Topology on the Closed Subsets of X

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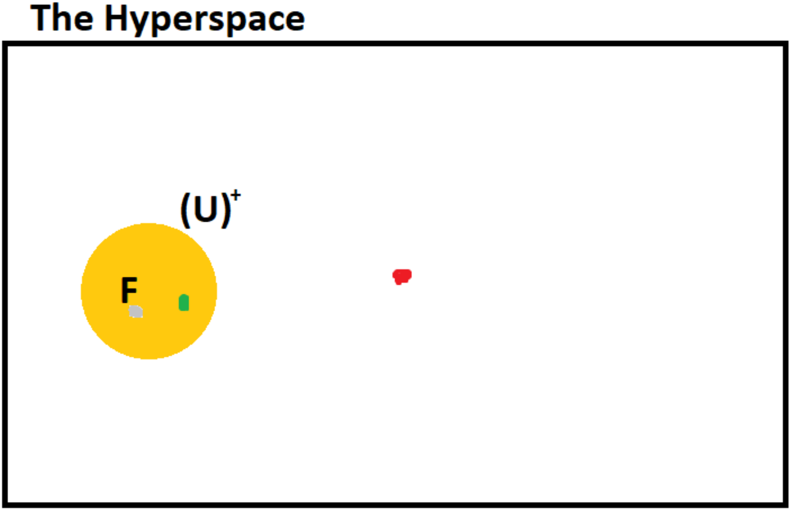

A Topological Connection

- $U \subseteq X$ is open if and only if $X \setminus U$ is a point in $\mathbb{F}(X)$.
- If $D \subseteq \mathbb{F}(X)$ is dense, then $\{X \setminus F : F \in D\}$ is an open cover of X.
- In fact, if $K \subseteq X$ is compact, then there is an $F \in D$ so that $K \subseteq X \setminus F$.

k-Covers

A non-trivial open cover $\mathscr{U}$ of X is a k**-cover** if for all compact $K \subseteq X$, there is a $U \in \mathscr{U}$ so that $K \subseteq U$.
$\mathcal{K}(X)$ is the collection of k-covers of X.

Proposition

- $\mathscr{U} \in \mathcal{K}(X)$ if and only if $\{X \setminus U : U \in \mathscr{U}\} \in \mathscr{D}_{\mathbb{F}(X)}$.
- $D \in \mathscr{D}_{\mathbb{F}(X)}$ if and only if $\{X \setminus F : F \in D\} \in \mathcal{K}(X)$.

Another Equivalence

- $S_1(\mathcal{K}(X), \mathcal{K}(X))$ if and only if $S_1(\mathscr{D}_{\mathbb{F}(X)}, \mathscr{D}_{\mathbb{F}(X)})$ (Mario, Kočinac, and Meccariello 2005).

Theorem (Caruvana and Holshouser 2021)

$G_1(\mathcal{K}(X), \mathcal{K}(X))$ is equivalent to $G_1(\mathscr{D}_{\mathbb{F}(X)}, \mathscr{D}_{\mathbb{F}(X)})$.

Proof.

- For $G_1(\mathcal{K}(X), \mathcal{K}(X)) \leq_{\mathrm{II}} G_1(\mathscr{D}_{\mathbb{F}(X)}, \mathscr{D}_{\mathbb{F}(X)})$ use $\varphi(F, n) = X \setminus F$.
- For $G_1(\mathscr{D}_{\mathbb{F}(X)}, \mathscr{D}_{\mathbb{F}(X)}) \leq_{\mathrm{II}} G_1(\mathcal{K}(X), \mathcal{K}(X))$, use $\psi(U, n) = X \setminus U$.

□

A Synchronized Turnless Translation Theorem

Corollary

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be collections with $\bigcup \mathcal{C} \subseteq \bigcup \mathcal{A}$ and $\bigcup \mathcal{D} \subseteq \bigcup \mathcal{B}$. Suppose there exists a bijection $\beta : \bigcup \mathcal{A} \to \bigcup \mathcal{B}$ so that:

- $A \in \mathcal{A}$ if and only if $\beta[A] \in \mathcal{B}$, and
- $C \in \mathcal{C}$ if and only if $\beta[C] \in \mathcal{D}$.

Then $G_1(\mathcal{A}, \mathcal{C})$ is equivalent to $G_1(\mathcal{B}, \mathcal{D})$.

To see that $G_1(\mathcal{K}(X), \mathcal{K}(X))$ is equivalent to $G_1(\mathscr{D}_{\mathbb{F}(X)}, \mathscr{D}_{\mathbb{F}(X)})$ use $\beta(U) = X \setminus U$.

Thanks for Listening