Selection Games on Continuous Functions

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Tkachuk's Question

Given an open set U of continuous functions $f : X \to \mathbb{R}$, when can we guarantee that there is an infinite subset of U which is closed discrete?

More generally, given a sequence of open sets U_n, when can we guarantee that there are choices $f_n \in U_n$ so that $\{f_n : n \in \omega\}$ is closed discrete?

Selection Principles/Games

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are collections of sets.

$S_1(\mathcal{A}, \mathcal{B})$

For every sequence $(A_n : n \in \omega)$ of sets from $\mathcal{A}$, there are $x_n \in A_n$ so that $\{x_n : n \in \omega\} \in \mathcal{B}$

$G_1(\mathcal{A}, \mathcal{B})$

$S_1(\mathcal{A}, \mathcal{B})$ can be turned into a two-player game $G_1(\mathcal{A}, \mathcal{B})$:

I	$A_0 \in \mathcal{A}$	$A_1 \in \mathcal{A}$	$\cdots$
II	$x_0 \in A_0$	$x_1 \in A_1$	$\cdots$

- Player II wins a given run of the game if $\{x_n : n \in \omega\} \in \mathcal{B}$.
- Player I wins a given run of the game if $\{x_n : n \in \omega\} \notin \mathcal{B}$.

Perfect Information Strategies

Fix a game $G_1(\mathcal{A}, \mathcal{B})$.

Perfect Information Strategy For Player I

I	$\sigma(\emptyset)$	$\sigma(x_0)$	$\sigma(x_0, x_1)$	$\cdots$
II	x_0	x_1	x_2	$\cdots$

σ is **winning** if it produces winning plays for I. If I has a winning strategy we write $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$.

Perfect Information Strategy for Player II

I	A_0	A_1	A_2	$\cdots$
II	$\tau(A_0)$	$\tau(A_0, A_1)$	$\tau(A_0, A_1, A_2)$	$\cdots$

τ is **winning** if it produces winning plays for II. If II has a winning strategy we write $\mathrm{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$.

Low Information Strategies

Pre-Determined for Player I

I	$\sigma(0)$	$\sigma(1)$	$\sigma(2)$	$\cdots$
II	x_0	x_1	x_2	$\cdots$

If I has a winning pre-determined strategy we write $\text{I} \uparrow_{pre} G_1(\mathcal{A}, \mathcal{B})$

Markov Strategy for Player II

I	A_0	A_1	A_2	$\cdots$
II	$\tau(A_0, 0)$	$\tau(A_1, 1)$	$\tau(A_2, 2)$	$\cdots$

If II has a winning Markov tactic we write $\text{II} \uparrow_{mark} G_1(\mathcal{A}, \mathcal{B})$.

Dual and Equivalent Games

Equivalent Games

Say $G_1(\mathcal{A}, \mathcal{C})$ and $G_1(\mathcal{B}, \mathcal{D})$ are **equivalent** if

- $\mathrm{I} \uparrow_{pre} G_1(\mathcal{A}, \mathcal{C}) \iff \mathrm{I} \uparrow_{pre} G_1(\mathcal{B}, \mathcal{D})$,
- $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{C}) \iff \mathrm{I} \uparrow G_1(\mathcal{B}, \mathcal{D})$,
- $\mathrm{II} \uparrow G_1(\mathcal{A}, \mathcal{C}) \iff \mathrm{II} \uparrow G_1(\mathcal{B}, \mathcal{D})$,
- $\mathrm{II} \uparrow_{mark} G_1(\mathcal{A}, \mathcal{C}) \iff \mathrm{II} \uparrow_{mark} G_1(\mathcal{B}, \mathcal{D})$.

Dual Games

Say $G_1(\mathcal{A}, \mathcal{C})$ and $G_1(\mathcal{B}, \mathcal{D})$ are **dual** if

- $\mathrm{I} \uparrow_{pre} G_1(\mathcal{A}, \mathcal{C}) \iff \mathrm{II} \uparrow_{mark} G_1(\mathcal{B}, \mathcal{D})$,
- $\mathrm{I} \uparrow G_1(\mathcal{A}, \mathcal{C}) \iff \mathrm{II} \uparrow G_1(\mathcal{B}, \mathcal{D})$,
- $\mathrm{II} \uparrow G_1(\mathcal{A}, \mathcal{C}) \iff \mathrm{I} \uparrow G_1(\mathcal{B}, \mathcal{D})$,
- $\mathrm{II} \uparrow_{mark} G_1(\mathcal{A}, \mathcal{C}) \iff \mathrm{I} \uparrow_{pre} G_1(\mathcal{B}, \mathcal{D})$.

A Strength Hierarchy

$$\mathrm{II} \uparrow_{mark} G_1(\mathcal{A},\mathcal{B}) \Rightarrow \mathrm{II} \uparrow G(\mathcal{A},\mathcal{B}) \Rightarrow \mathrm{I} \not\uparrow G_1(\mathcal{A},\mathcal{B}) \Rightarrow \mathrm{I} \not\uparrow_{pre} G_1(\mathcal{A},\mathcal{B})$$

$$\mathrm{I} \not\uparrow_{pre} G_1(\mathcal{A},\mathcal{B}) \iff S_1(\mathcal{A},\mathcal{B})$$

Notation

- $\mathcal{T}_X$ is the collection of open subsets of X,
- Ω_X is the collection of all open covers $\mathcal{U}$ with $X \notin \mathcal{U}$ and so that whenever $F \subseteq X$ is finite, there is a $U \in \mathcal{U}$ so that $F \subseteq U$,
- for $F \subseteq X$ finite, $\mathcal{N}(F)$ is the collection of open sets U with $F \subseteq U$,
- $\mathcal{N}[X^{<\omega}] = \{\mathcal{N}(F) : F \subseteq X \text{ is finite}\}$,
- $C_p(X)$ is the space of continuous functions $f : X \to \mathbb{R}$ with the topology of point-wise convergence. Its basis is sets of the form

$$[f; F, \varepsilon] = \{g \in C_p(X) : (\forall x \in F)[|f(x) - g(x)| < \varepsilon]\}.$$

  where $F \subseteq X$ is finite,
- $\Omega_{X,x}$ is all sets A with $x \in \overline{A} \setminus A$,
- CD_X is the set of all closed discrete subsets of X

Selection Games We Play

- $G_1(\Omega_X, \Omega_X)$ - A modification of the Rothberger game
  Do nice covers have nice countable subcovers?
- $G_1(\mathcal{N}[X^{<\omega}], \neg\Omega_X)$ - A modification of the point-open game
  What does it take to fill out the space?
- $G_1(\mathcal{N}_{C_p(X)}(\mathbf{0}), \neg\Omega_{C_p(X),\mathbf{0}})$ - Gruenhage's game
  How do neighborhoods of $\mathbf{0}$ funnel functions towards $\mathbf{0}$?
- $G_1(\mathcal{T}_{C_p(X)}, \mathrm{CD}_{C_p(X)})$ - Tkachuk's game
  Do open sets of functions contain closed discrete subsets?
- $G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$ (The countable fan tightness game)
  Do sequences witness closure?

Answering Tkachuk's Question

Theorem (Tkachuk 2018 and Clontz-H. 2019)

Suppose X is a $T_{3.5}$-space. Then

- $G_1(\mathcal{N}[X^{<\omega}], \neg\Omega_X)$, $G_1(\mathcal{N}_{C_p(X)}(\mathbf{0}), \neg\Omega_{C_p(X),\mathbf{0}})$, and $G_1(\mathcal{T}_{C_p(X)}, \mathrm{CD}_{C_p(X)})$ are equivalent.
- $G_1(\Omega_X, \Omega_X)$ and $G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$ are equivalent.
- These groups are dual to each other.
- $\mathrm{I} \uparrow_{pre} G_1(\mathcal{T}_{C_p(X)}, \mathrm{CD}_{C_p(X)})$ if and only if X is countable if and only if $C_p(X)$ is first countable.
- $S_1(\mathcal{T}_{C_p(X)}, \mathrm{CD}_{C_p(X)})$ if and only if X is uncountable.

Which Open Sets?

What is the answer to Tkachuk's question if instead of $C_p(X)$, we look at $C_k(X)$: continuous functions $f : X \to \mathbb{R}$ with the topology of uniform convergence on compact sets?

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Basic Open Sets in $C_k(X)$

For $f : X \to \mathbb{R}$ continuous, $K \subseteq X$ compact, and $\varepsilon > 0$, set

$$[f; K, \varepsilon] = \{g \in C_k(X) : (\forall x \in K)[|f(x) - g(x)| < \varepsilon]\}.$$

These form a basis for the topology on $C_k(X)$.

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- for $F \subseteq X$ finite, $\mathcal{N}(F)$ is the collection of open sets U with $F \subseteq U$,
- $\mathcal{N}[X^{<\omega}] = \{\mathcal{N}(F) : F \subseteq X \text{ is finite}\}$

More Notation

- $\mathcal{K}(X)$ is the set of compact subsets of X,
- $\mathcal{K}_X$ is the collection of all open covers $\mathcal{U}$ with $X \notin \mathcal{U}$ and so that if $K \subseteq X$ is compact, then there is a $U \in \mathcal{U}$ so that $K \subseteq U$,
- for $K \subseteq X$ compact, $\mathcal{N}(K)$ is the collection of open sets U with $K \subseteq U$,
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Answering Tkachuk's Question Again

Theorem (Caruvana-H. 2019)

Suppose X is a $T_{3.5}$-space. Then

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Answering Tkachuk's Question Again

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- $G_1(\mathcal{K}_X, \mathcal{K}_X)$ and $G_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$ are equivalent.
- These groups are dual to each other.
- $\mathrm{I} \uparrow_{pre} G_1(\mathcal{T}_{C_k(X)}, \mathrm{CD}_{C_k(X)})$ if and only if X is hemicompact if and only if $C_k(X)$ is first countable.
- $S_1(\mathcal{T}_{C_k(X)}, \mathrm{CD}_{C_k(X)})$ if and only if X is not hemicompact.

Note: Arens, Aurichi, Dias, Kočinac, Osipov, Scheepers, and others have shown similar connections between X, $C_p(X)$, and $C_k(X)$.

Hemicompact?

Definition

X is **hemicompact** if there is a sequence of compact sets K_n so that $X = \bigcup_n K_n$ and whenever $L \subseteq X$ is compact, there is an n so that $L \subseteq K_n$.

Hemicompact is strictly stronger than σ-compact.

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Hemicompact is strictly stronger than σ-compact.

Questions

1. How can we use these games to assert that X is σ-compact?
2. What's really happening behind the scenes that allows these results to be so similar?

The Big Picture

The following correspondences are really what matters here:

- $F \subseteq X$ finite generates $[\mathbf{0}; F, 2^{-n}]$
- The range of a choice function for $\mathcal{N}[X^{<\omega}]$ forms a cover $\mathcal{U} \in \Omega_X$
- Given a basic open set $U = [f; F, \varepsilon] \subseteq C_p(X)$ and another open set $V \subseteq X$ with $F \subseteq X$, there is a function g so that $g \in U$ and $g[X \setminus V] = \{n\}$
- The range of a choice function for $\mathcal{N}_{C_p(X)}(\mathbf{0})$ forms a set $F \in \Omega_{C_p(X),\mathbf{0}}$.

All of this happens in the compact-open setting as well.

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- The range of a choice function for $\mathcal{N}_{C_p(X)}(\mathbf{0})$ forms a set $F \in \Omega_{C_p(X),\mathbf{0}}$.

All of this happens in the compact-open setting as well.

Goals

1. Generalize these objects and their relationships.
2. Build a device to show that games are equivalent/dual using only the relationships between the objects.

Even More Notation

Suppose $\mathcal{A} \subseteq \mathcal{P}(X)$.

- $\mathcal{O}(X, \mathcal{A})$ is the collection of all open covers $\mathcal{U}$ with $X \notin \mathcal{U}$ and so that if $A \in \mathcal{A}$, then there is a $U \in \mathcal{U}$ so that $A \subseteq U$.
- For $A \in \mathcal{A}$, $\mathcal{N}(A)$ is the set of all open sets U with $A \subseteq U$.
- $\mathcal{N}[\mathcal{A}] = \{\mathcal{N}(A) : A \in \mathcal{A}\}$.
- $C_{\mathcal{A}}(X)$ is the set of continuous functions $f : X \to \mathbb{R}$ with topology generated by sets of the form

$$[f; A, \varepsilon] = \{g \in C_{\mathcal{A}}(X) : (\forall x \in A)[|f(x) - g(x)| < \varepsilon]\}$$

Generalizing Hemicompactness

The following is inspired by Gartside's work with the Tukey order and Telgársky's work on the point-open game.

Definition

Suppose $\mathbb{P} = (P, \leq)$ is a partial order and that $\mathcal{A}, \mathcal{B} \subseteq \mathbb{P}$. We say

$$\text{cof}(\mathcal{A}; \mathcal{B}, \leq) = \omega$$

if there is a sequence of $A_n \in \mathcal{A}$ so that for all $B \in \mathcal{B}$, there is an n so that $B \leq A_n$.

- $\text{cof}(X^{<\omega}; X^{<\omega}, \subseteq) = \omega$ if and only if X is countable.
- $\text{cof}(\mathcal{K}(X); X^{<\omega}, \subseteq) = \omega$ if and only if X is σ-compact.
- $\text{cof}(\mathcal{K}(X); \mathcal{K}(X), \subseteq) = \omega$ if and only if X is hemicompact.

Some Conditions on $\mathcal{A}$

Suppose X is $T_{3.5}$ and $\mathcal{A} \subseteq \mathcal{P}(X)$.

- $\mathcal{A}$ is an **ideal base** if whenever $A_1, A_2 \in \mathcal{A}$, there is an $A_3 \in \mathcal{A}$ so that $A_1 \cup A_2 \subseteq A_3$.
- X is $\mathcal{A}$-**normal** if whenever $A \in \mathcal{A}$, U is open, and $A \subseteq U$, there is a continuous function $f : X \to \mathbb{R}$ so that $f[A] = \{0\}$ and $f[X \setminus U] = \{1\}$.
- $\mathcal{A}$ is $\mathbb{R}$-**bounded** if for all $A \in \mathcal{A}$ and $f : X \to \mathbb{R}$ continuous, $f[A]$ is bounded.

A General Answer to Tkachuk's Question

Theorem (Caruvana-H. 2019)

Let X be a $T_{3.5}$-space and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ be ideal bases. Suppose $\mathcal{A}$ consists of closed sets, X is $\mathcal{A}$-normal, and $\mathcal{B}$ is $\mathbb{R}$-bounded.

- $G_1(\mathcal{N}[\mathcal{A}], \neg\mathcal{O}(X, \mathcal{B}))$, $G_1(\mathcal{N}_{C_\mathcal{A}(X)}(\mathbf{0}), \neg\Omega_{C_\mathcal{B}(X),\mathbf{0}})$, and $G_1(\mathcal{T}_{C_\mathcal{A}(X)}, \mathrm{CD}_{C_\mathcal{B}(X)})$ are equivalent.
- $G_1(\mathcal{O}(X, \mathcal{A}), \mathcal{O}(X, \mathcal{B}))$ and $G_1(\Omega_{C_\mathcal{A}(X),\mathbf{0}}, \Omega_{C_\mathcal{B}(X),\mathbf{0}})$ are equivalent.
- These groups are dual to each other.
- $\mathrm{I} \uparrow_{pre} G_1(\mathcal{T}_{C_\mathcal{A}(X)}, \mathrm{CD}_{C_\mathcal{B}(X)})$ if and only if

$$\mathrm{cof}(\mathcal{A} \times \omega; \mathcal{B} \times \omega, \subseteq) = \mathrm{cof}(\mathcal{N}_{C_\mathcal{A}(X)}(\mathbf{0}); \mathcal{N}_{C_\mathcal{B}(X)}(\mathbf{0}), \supseteq) = \omega$$

Corollary (Caruvana-H. 2019)

$S_1(\mathcal{T}_{C_k(X)}, \mathrm{CD}_{C_p(X)})$ if and only if X is not σ-compact.

Showing Games Are Dual

Clontz has developed a general tool for showing that two games are dual. We used this throughout.

- Let choice($\mathcal{A}$) be the set of functions $C : \mathcal{A} \to \bigcup \mathcal{A}$ so that $C(A) \in A$.
- $\mathcal{A}$ is a **selection basis** for $\mathcal{B}$ if whenever $B \in \mathcal{B}$, there is an $A \in \mathcal{A}$ so that $A \subseteq B$
- $\mathcal{R}$ is a **reflection** of $\mathcal{A}$ if

$$\{\text{ran}(C) : C \in \text{choice}(\mathcal{R})\}$$

is a selection basis for $\mathcal{A}$.

Theorem (Clontz 2018)

If $\mathcal{R}$ is a reflection of $\mathcal{A}$, then $G_1(\mathcal{A}, \mathcal{B})$ and $G_1(\mathcal{R}, \neg\mathcal{B})$ are dual.

Showing Games Are Equivalent

Definition

Say that $G_1(\mathcal{A},\mathcal{C}) \leq_{\text{II}} G_1(\mathcal{B},\mathcal{D})$ if

- $\text{I} \not\uparrow_{pre} G_1(\mathcal{A},\mathcal{C}) \implies \text{I} \not\uparrow_{pre} G_1(\mathcal{B},\mathcal{D})$,
- $\text{I} \not\uparrow G_1(\mathcal{A},\mathcal{C}) \implies \text{I} \not\uparrow G_1(\mathcal{B},\mathcal{D})$,
- $\text{II} \uparrow G_1(\mathcal{A},\mathcal{C}) \implies \text{II} \uparrow G_1(\mathcal{B},\mathcal{D})$, and
- $\text{II} \uparrow_{mark} G_1(\mathcal{A},\mathcal{C}) \implies \text{II} \uparrow_{mark} G_1(\mathcal{B},\mathcal{D})$.

This is transitive and if $G_1(\mathcal{A},\mathcal{C}) \leq_{\text{II}} G_1(\mathcal{B},\mathcal{D})$ and $G_1(\mathcal{B},\mathcal{D}) \leq_{\text{II}} G_1(\mathcal{A},\mathcal{C})$, then the games are equivalent.

Showing Games Are Equivalent

Theorem (Caruvana-H. 2019)

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be collections. Suppose there are functions

- $\overleftarrow{T}_{\mathrm{I},n} : \mathcal{B} \to \mathcal{A}$ and
- $\overrightarrow{T}_{\mathrm{II},n} : \bigcup \mathcal{A} \times \mathcal{B} \to \bigcup \mathcal{B}$

so that

1. if $x \in \overleftarrow{T}_{\mathrm{I},n}(B)$, then $\overrightarrow{T}_{\mathrm{II},n}(x, B) \in B$ and
2. if $x_n \in \overleftarrow{T}_{\mathrm{I},n}(B_n)$ for all n, then

$$\{x_n : n \in \omega\} \in \mathcal{C} \implies \{\overrightarrow{T}_{\mathrm{II},n}(x_n, B_n) : n \in \omega\} \in \mathcal{D}$$

Then $G_1(\mathcal{A}, \mathcal{C}) \leq_{\mathrm{II}} G_1(\mathcal{B}, \mathcal{D})$.

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Then $G_1(\mathcal{A}, \mathcal{C}) \leq_{\mathrm{II}} G_1(\mathcal{B}, \mathcal{D})$.

(We also generalized results from Tkachuk and Pawlikowski to go from full strategies to limited strategies)

Thanks for Listening