Selection Games: Equivalence and Duality

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Selection Games

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are collections of sets.

$G_1(\mathcal{A}, \mathcal{B})$ (Scheepers 1996) (Menger,Hurewicz 1924)

At round n, player I plays $A_n \in \mathcal{A}$ and player II plays $x_n \in \mathcal{A}_n$.

I	$A_0 \in \mathcal{A}$	$A_1 \in \mathcal{A}$	$\cdots$
II	$x_0 \in A_0$	$x_1 \in A_1$	$\cdots$

- Player II wins a given run of the game if $\{x_n : n \in \mathbb{N}\} \in \mathcal{B}$.
- Player I wins a given run of the game if $\{x_n : n \in \mathbb{N}\} \notin \mathcal{B}$.

Examples of Selection Games

- The Rothberger game (Rothberger 1938):

$$G_1(\text{“covers”}, \text{“covers”})$$

- The Strong Countable Fan Tightness game (Arhangel’skii 1986):

$$G_1(\text{“sets clustering to } x_0\text{”}, \text{“sequences clustering to } x_0\text{”})$$

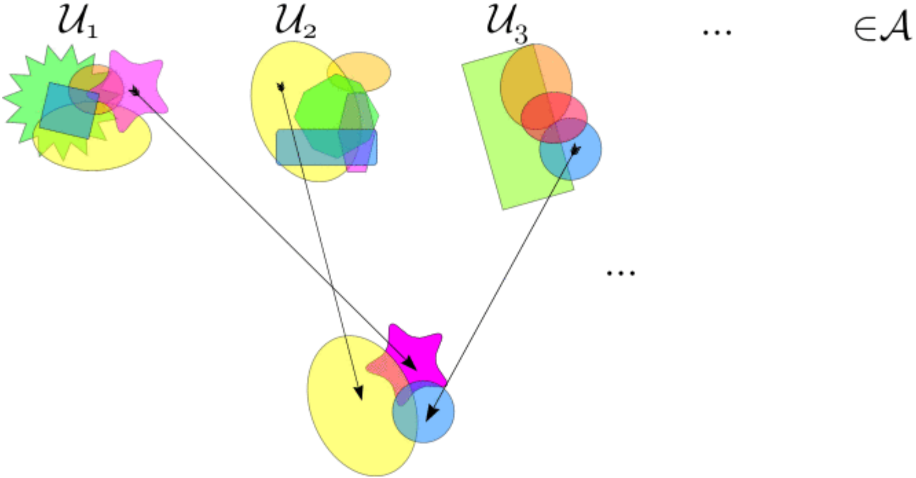
- The Point-Open game (Galvin 1978):

$$G_1(\text{“neighborhood systems”}, \text{“covers”})$$

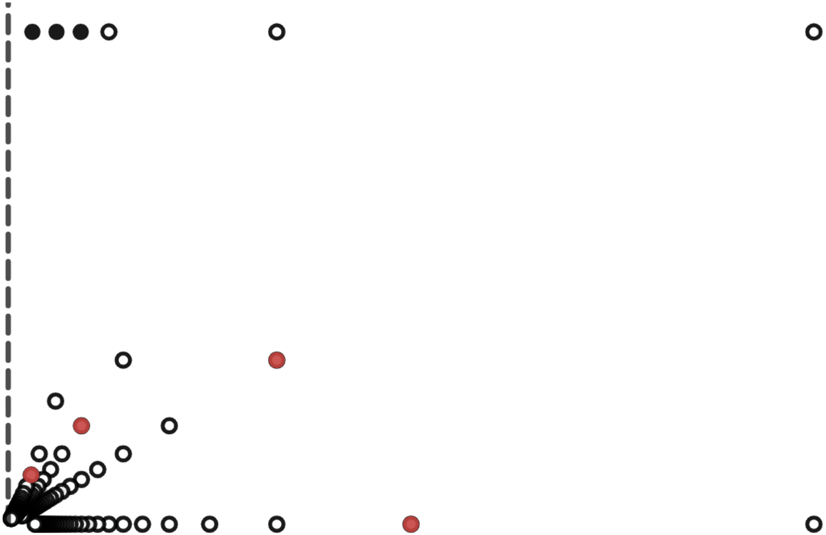
- The Closed-Discrete Game (Tkachuk 2017):

$$G_1(\text{“open sets”}, \text{“closed discrete sets”})$$

The Rothberger Game



The Strong Countable Fan Tightness Game



A Strength Hierarchy

For the Rothberger game:

II has a strong winning strategy	$\iff$	the space is countable
$\Downarrow$		
II has a weak winning strategy		
$\Downarrow$		
I has no strong winning strategy		
$\Downarrow$		
I has no weak winning strategy	$\iff$	can diagonally make covers

Some Terminology

- An open cover of X is an ω-cover if every finite subset of X is contained in some open set from the cover.
- An open cover of X is an k-cover if every compact subset of X is contained in some open set from the cover.
- $C_p(X)$ is the collection of continuous functions from X to $\mathbb{R}$ with the topology of point-wise convergence.
- $C_k(X)$ is the collection of continuous functions from X to $\mathbb{R}$ with the topology of uniform covergence on compact sets.

Connections on $C_p(X)$

Suppose X is Hausdorff and completely regular.

1. The ω, ω-Rothberger game on X and the strong countable fan tightness game on $C_p(X)$ are equivalent (Sakai 1988) (Clontz and H. 2019).
2. The finite-ω game on X and the closed discrete game on $C_p(X)$ are equivalent (Tkachuk 2017) (Clontz and H. 2019).
3. The strong countable fan tightness game on $C_p(X)$ and the closed discrete game on $C_p(X)$ are dual (Galvin 1978) (Clontz and H. 2019).
4. II has a strong strategy in ω, ω-Rothberger game on X if and only if X is countable.

Connections on $C_k(X)$

Suppose X is Hausdorff and completely regular.

1. The k, k-Rothberger game on X and the strong countable fan tightness game on $C_k(X)$ are equivalent (Kocinac 2003) (Caruvana and H. 2019).
2. The compact-k game on X and the closed discrete game on $C_k(X)$ are equivalent (Caruvana and H. 2019).
3. The strong countable fan tightness game on $C_k(X)$ and he closed discrete game on $C_k(X)$ are dual (Caruvana and H. 2019).
4. II has a strong strategy in k, k-Rothberger game on X if and only if the hyperspace of compact subsets of X is a countable union of compact sets.

Cross Topology Connections

Suppose X is Hausdorff and completely regular.

1. The k, ω-Rothberger game on X and the strong countable fan tightness game from $C_k(X)$ to $C_p(X)$ are equivalent (Caruvana and H. 2020).
2. The compact-ω game on X and the closed discrete game from $C_k(X)$ to $C_p(X)$ are equivalent (Caruvana and H. 2020).
3. The equivalent duality statement holds.
4. II has a strong strategy in k, ω-Rothberger game on X if and only if X is a countable union of compact sets.

Thanks for Listening