An Adaptation of the Vietoris Topology for Ordered Compact Sets

Jared Holshouser (joint work with C. Caruvana

Unaffiliated

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Inspiration

- Teaching combinatorics in discrete math.
- Proving results about covering properties for $\bigcup_n X^n$, the hyperspace of finite sets, and the hyperspace of compact sets.

This led to two questions:

- How do we topologize the space of ordered compact sets?
- What properties (covering and otherwise) does this space have?

The Vietoris Topology

The **Vietoris Topology** on the hyperspace of closed subsets of X is generated by basic open sets of the form

$$[U; V_1, \cdots, V_n] = \{F : F \subseteq U \text{ and } F \cap V_j \neq \emptyset \text{ for } 1 \leq j \leq n\}$$

where $U, V_1, \cdots, V_n \subseteq X$ are open.

Let $\mathbb{K}(X)$ denote the Vietoris topology on the hyperspace of compact sets and $\mathbb{F}(X)$ denote the Vietoris topology on the hyperspace of finite sets.

The Vietoris Power

Let X be a space and κ be a cardinal. For $U \subseteq X$, $\Lambda \in [\kappa]^{<\omega}$ and $V : \Lambda \to \mathcal{P}(X)$, define

$$[U; \Lambda, V] = \{f \in X^\kappa : [\forall \alpha \in \Lambda (f(\alpha) \in V_\alpha)] \wedge \mathrm{img}(f) \subseteq U\}$$

The sets of the form $[U; \Lambda, V]$ form a basis for a topology when U, V range over the open subsets of X.

We denote this topology on X^κ as $V(X^\kappa)$ and refer to it as the **Vietoris power of order** κ.

Note: This space was first studied by Piotrowski, Roslanowski, and Scott in 1983 under the name pinched-cube topology.

Special Neighborhoods

There are two types of neighborhoods in the Vietoris power that deserve special attention.

- tubes: $[U; \emptyset, \emptyset] = U^\kappa$
- cones: $[X; \Lambda, V] = \bigcap_{\alpha \in \Lambda} \pi_\alpha^{-1}(V_\alpha)$

Ordered Hyperspaces

Fix a space X and a cardinal κ.

- $\mathbb{K}(X, \text{ord}, \kappa) = \{f \in X^\kappa : \text{img}(f) \text{ is compact}\}$.
- $\mathbb{K}(X, \text{ord}) = \mathbb{K}(X, \text{ord}, \sup\{\#K : K \in K(X)\} + \omega)$
- $\mathbb{F}(X, \text{ord}, \kappa) = \{f \in X^\kappa : \text{img}(f) \text{ is finite}\}$.
- $\mathbb{F}(X, \text{ord}) = \mathbb{F}(X, \text{ord}, \omega)$

Note: $\mathbb{F}(X, \text{ord}) \neq \bigcup_n [X]^n$, so we don't have an exact analogue. However, uncountable disjoint unions come with undesirable covering properties.

Compactness

$V(2^\omega)$ is not compact.

Consider the cover formed by $\{0\}^\omega$ and $V_n = \{b \in 2^\omega : b_n = 1\}$.

Proposition

Consider a space X and a cardinal κ. Let $\mathscr{F} \subseteq \mathbb{K}(X, \mathrm{ord}, \kappa)$ be so that $\bigcup\{\mathrm{img}(f) : f \in \mathscr{F}\}$ is not compact. Then $\mathscr{F}$ is not compact.

Corollary

Let κ be a cardinal.

- If $\mathscr{F} \subseteq \mathbb{K}(D(\kappa), \mathrm{ord})$ is so that $\bigcup_{f \in \mathscr{F}} \mathrm{img}(f)$ is infinite, or
- If $\mathscr{F} \subseteq \mathbb{K}(\kappa, \mathrm{ord})$ is so that $\bigcup_{f \in \mathscr{F}} \mathrm{img}(f)$ is unbounded, or
- If $\mathscr{F} \subseteq \mathbb{K}(\mathbb{R}, \mathrm{ord})$ is so that $\bigcup_{f \in \mathscr{F}} \mathrm{img}(f)$ is unbounded,

then $\mathscr{F}$ is not compact.

Density

Lemma

For any infinite cardinal κ, $d\left(\mathsf{V}\left(D(\kappa)^{\exp\kappa}\right)\right) \leq \kappa$.
In particular, there is a dense subset of $\mathsf{V}\left(D(\kappa)^{\exp\kappa}\right)$ of cardinality κ consisting of functions with finite range. Such a dense set is thus contained in $\mathbb{F}(D(\kappa), \mathrm{ord}, 2^\kappa) = \mathbb{K}(D(\kappa), \mathrm{ord}, 2^\kappa)$.

A Hewitt-Marczewski-Pondiczery Style Theorem

Suppose $\kappa \leq 2^{d(X)}$ for a space X. Then $d(\mathbb{F}(X, \mathrm{ord}, \kappa)) \leq d(X)$.
Consequently, $d(\mathbb{K}(X, \mathrm{ord}, \kappa)) \leq d(X)$ and $d(\mathsf{V}(X^\kappa)) \leq d(X)$.

Comparison to Other Products

Proposition

For any space X with at least two points and any infinite cardinal κ, $\mathsf{V}(X^\kappa)$ has non-isolated points. Moreover, the set of non-constant functions of $\mathsf{V}(X^\kappa)$ is a crowded subspace.

Proposition

The topology of $\mathsf{V}(X^\kappa)$ is finer than the Tychonoff topology on X^κ, coarser than the box topology on X^κ, and it need not be homeomorphic to either.

Note: The Vietoris topology can be incomparable with Bell's uniform box topology.

The Vietoris Power on Finite Sets

Theorem

For any natural number $n \geq 2$, $\mathsf{V}(n^\omega)$ has the following properties:

1. $\mathsf{V}(n^\omega)$ is second-countable.
2. $\mathsf{V}(n^\omega)$ is not homogeneous.
3. $\mathsf{V}(n^\omega)$ is zero-dimensional.
4. $\mathsf{V}(n^\omega)$ is σ-compact.
5. $\mathsf{V}(n^\omega)$ is locally compact.

Consequently, $\mathsf{V}(n^\omega)$ is separable, metrizable, Baire, and not a topological group.

Ordered Compact Subsets of Finite Sets

Theorem

The space $\mathbb{K}(\omega, \mathrm{ord})$ has the following properties:

1. $\mathbb{K}(\omega, \mathrm{ord})$ is second-countable.
2. $\mathbb{K}(\omega, \mathrm{ord})$ is not homogeneous.
3. $\mathbb{K}(\omega, \mathrm{ord})$ is zero-dimensional.
4. $\mathbb{K}(\omega, \mathrm{ord})$ is σ-compact.
5. $\mathbb{K}(\omega, \mathrm{ord})$ is locally compact.
6. $\mathbb{K}(\omega, \mathrm{ord})$ is completely metrizable.

Consequently, $\mathbb{K}(\omega, \mathrm{ord})$ is separable, Baire, not a topological group, and embeds as a subspace of 2^ω.

The Vietoris Power of the Naturals

Theorem

The space $\mathsf{V}(\omega^\omega)$ has the following properties:

1. It is not homogenous, locally compact, scattered, a GO-space, or Menger.
2. It is separable, first-countable, zero-dimensional, and Baire.
3. Every function with finite range has a local basis consisting of compact sets.
4. Functions with infinite range do not have compact neighborhoods.
5. $w(\mathsf{V}(\omega^\omega)) = s(\mathsf{V}(\omega^\omega)) = e(\mathsf{V}(\omega^\omega)) = \mathfrak{c}$.
6. There is a continuous bijection $\mathsf{V}(\omega^\omega) \to \mathsf{V}(\omega^\omega)^2$.

Consequently, $\mathsf{V}(\omega^\omega)$ is completely regular, not σ-compact, not metrizable, not hereditarily separable, and not hereditarily Lindelöf.

For the Pi Base Fans

	$b(\omega^\omega)$	ω^ω	$V(\omega^\omega)$	$V(\omega^\omega)'$	$\mathbb{R}_\ell$	$\mathbb{R}_{rs}$
first-countable	✓	✓	✓	✓	✓	✓
second-countable	✗	✓	✗	✗	✗	✗
separable	✗	✓	✓	✓	✓	✓
countable spread	✗	✓	✗	✗	✓	✗
zero-dimensional	✓	✓	✓	✓	✓	✓
Baire	✓	✓	✓	✓	✓	✓
crowded	✗	✓	✗	✓	✓	✗
scattered	✓	✗	✗	✗	✗	✓
GO-space	✓	✓	✗	✗	✓	✗
homogeneous	✓	✓	✗	✗	✓	✗
locally compact	✓	✗	✗	✗	✗	✓
Menger	✗	✗	✗	✗	✗	✗
Lindelöf	✗	✓	✗	✗	✓	✗
normal	✓	✓	✗	✗	✓	✗
countable extent	✗	✓	✗	✗	✓	✗

Table 1. Comparison of particular topologies on the continuum.

Covering Properties for $\mathbb{F}(X, \text{ord})$

Theorem

If $\mathbb{F}(X, \text{ord})$ is Rothberger/Menger, then X is ω-Rothberger/ω-Menger.

Proposition

For any T_1 space X, $\mathbb{F}(X, \text{ord})$ is Rothberger if and only if X is a singleton.

Example

$X = \{0, 1\}$ is ω-Rothberger, but $\mathbb{F}(X, \text{ord})$ is not Rothberger.

A Helpful Proposition

Definition

Let X and Y be topological spaces. Then a continuous $f : X \to Y$ is a *compact covering map* if whenever $L \subseteq Y$ is compact, there is a compact $K \subseteq X$ so that $f[K] = L$. If there is a compact covering map from X to Y, we say that X *is a compact covering of* Y.

Proposition

If X is a compact covering of Y and X is k-Rothberger/k-Menger, then Y is k-Rothber/k-Menger.

Covering Properties for $\mathbb{K}(X, \mathrm{ord})$

Theorem

- If $\mathbb{K}(X, \mathrm{ord})$ is Rothberger/Menger, then X is k-Rothberger/k-Menger.
- If $\mathbb{K}(X, \mathrm{ord})$ is k-Rothberger/k-Menger, then X is k-Rothberger/k-Menger.

Example

$\mathbb{K}(\mathbb{R}, \mathrm{ord})$ is not Lindelöf. Thus X can be k-Rothberger while $\mathbb{K}(X, \mathrm{ord})$ is not even k-Menger.
The offending cover is

$$\mathscr{U} := \{(-1,1)^{\mathfrak{c}}\} \cup \{\{f \in \mathbb{K}(\mathbb{R}, \mathrm{ord}) : |f(\alpha)| > 1/2\} : \alpha \in \mathfrak{c}\}.$$

Questions

- We proved that if α is an uncountable ordinal, then $\mathbb{K}(\alpha, \mathrm{ord})$ is not Lindelöf. We also know that $\mathbb{K}(\omega, \mathrm{ord})$ is Lindelöf (σ-compact actually). Is it true that $\mathbb{K}(\alpha, \mathrm{ord})$ is Lindelöf when $\omega < \alpha < \omega_1$?
- Is it true that when X is ω-Menger, $\mathbb{F}(X, \mathrm{ord})$ is Menger?

Thanks!

Thanks for Listening

You can find the paper at Applied General Topology, Vol. 27, No. 1