# Clustering Properties of CUSCO Functions

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# CUSCO Origins - Convex Optimization

In the field of convex optimization, they deal with mins and maxes of functions which are not necessarily differentiable. For a quick example, we know that $|x|$ has a min at $x = 0$, even though the derivative doesn't exist.

One way to analytically find this minimum is to take the closure of the graph of the derivative, take the convex hull of each vertical section, and then check if 0 is in the resulting section.

# CUSCO Maps - Definition

## USCO Maps

A set valued function $\Phi : X \to \mathcal{P}(Y)$ is upper semicontinuous if for every open $V \subseteq Y$,

$$\Phi^{\leftarrow}[V] = \{x \in X : \Phi(x) \subseteq V\}$$

is open.
$\Phi$ is called an **usco** map if it is upper semicontinuous and $\Phi(x)$ is compact for all $x \in X$.

## CUSCO Maps

An usco map $\Phi$ is called a **cusco** map if $\Phi(x)$ is convex for all $x \in X$.
A cusco map $\Phi$ is **minimal** if the graph of $\Phi$ is minimal with respect to $\subseteq$ in the set of cusco maps.
$\mathrm{MC}(X, Y)$ is the set of minimal cusco maps from $X$ to $Y$.

# Topologies on CUSCO Maps

Fix a topological space $X$. We define the topology on the cusco maps from $X$ to $\mathbb{R}$ as a uniform topology.

- First, for $Y \subseteq \mathbb{R}$ and $\varepsilon > 0$, let $\mathbb{B}(Y, \varepsilon) = \bigcup_{y \in Y} B(y, \varepsilon)$,
- Then, for $A \subseteq X$ and $\varepsilon > 0$, let $\mathbf{W}(A, \varepsilon)$ be

$$\{(\Phi, \Psi) : (\forall x \in A)[\Phi(x) \subseteq \mathbb{B}(\Psi(x), \varepsilon) \text{ and } \Psi(x) \subseteq \mathbb{B}(\Phi(x), \varepsilon)]\}$$

- Finally, $[\Phi; A, \varepsilon] = \mathbb{W}(A, \varepsilon)[\Phi]$.
- $\mathrm{MC}_p(X)$ is the topology on $\mathrm{MC}(X, \mathbb{R})$ generated by the $[\Phi; A, \varepsilon]$, where $A$ ranges over the finite subsets of $X$.
- $\mathrm{MC}_k(X)$ is the topology on $\mathrm{MC}(X, \mathbb{R})$ generated by the $[\Phi; A, \varepsilon]$, where $A$ ranges over the compact subsets of $X$.

# A Clever Characterization

In 2014, Holá and Holý proved the following clever characterization of minimal cusco maps.

**Theorem(Holá, Holý 2014)**

Suppose $Y$ is a Hausdorff locally convex linear space in which the closed convex hull of a compact set is compact and that $\Phi : X \to \mathcal{P}^+(Y)$. Then the following are equivalent.

1. $\Phi \in \mathrm{MC}(X, Y)$,
2. There is a selection $f$ of $\Phi$ which is subcontinuous, quasicontinuous, and $\Phi = \check{f}$.
3. There exists a densely defined selection $f$ of $\Phi$ which is subcontinuous, quasicontinuous, and $\Phi = \check{f}$.

# Unpacking the Theorem

- If $f : X \to \mathbb{R}$, then $\overline{f} : X \to \mathcal{P}(\mathbb{R})$ is the closure of the graph of $f$ and $\check{f}(x)$ is the closure of the convex hull of $\overline{f}(x)$.
- Suppose $D \subseteq X$ is dense and $f : D \to \mathbb{R}$. Then $f$ is **subcontinuous** if for every $x \in X$ and every net $x_\lambda \to x$, $\langle f(x_\lambda) \rangle$ has an accumulation point.
- A function $f : X \to \mathbb{R}$ is **quasicontinuous** if for every open $V \subseteq \mathbb{R}$, $f^{-1}[V]$ is semi-open.
- $A \subseteq X$ is **semi-open** if $A \subseteq \mathrm{cl}(\mathrm{int}(A))$.

# Selection Principles (Menger, Hurewicz 1924) (Scheepers 1996)

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are collections.

**$S_1(\mathcal{A}, \mathcal{B})$**

$S_1(\mathcal{A}, \mathcal{B})$ means that for all sequences $A_n$ consisting of elements of $\mathcal{A}$, there are choices $x_n \in A_n$ so that $\{x_n : n \in \omega\} \in \mathcal{B}$.

Let $\mathcal{O}(X)$ denote the open covers of $X$. A basic example of a selection principle is $S_1(\mathcal{O}(X), \mathcal{O}(X))$, a generalization of compactness that we refer to as **Rothberger**.

# Selection Games

- We can view the selection principle $S_1(\mathcal{A}, \mathcal{B})$ as a game process wherein player I plays sets $A_n$ and player II responds with $x_n \in A_n$.
- Player II wins if $\{x_n : n \in \omega\} \in \mathcal{B}$. We call this game $G_1(\mathcal{A}, \mathcal{B})$. Otherwise player I wins.
- In this game framework it's natural to impose information conditions on the players. These create a hierarchy of statements. In the Rothberger case, this looks like

$$X \text{ is ctbl } \rightarrow \text{ II wins } G_1(\mathcal{O}, \mathcal{O}) \rightarrow \text{ I doesn't win } G_1(\mathcal{O}, \mathcal{O}) \rightarrow S_1(\mathcal{O}, \mathcal{O})$$

# Comparing Games

## Definition

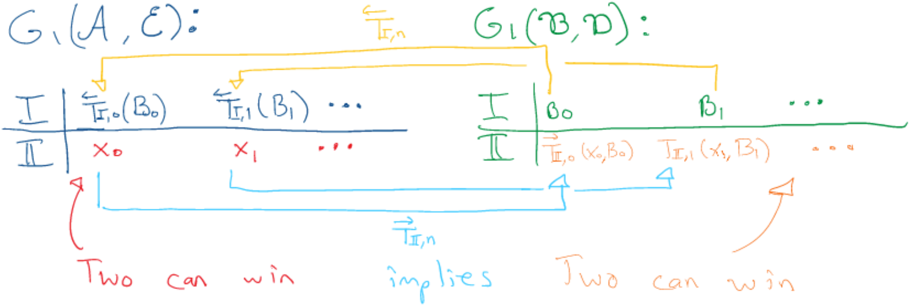
Define $G_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_1(\mathcal{B}, \mathcal{D})$ as the conjunction of the following implications.

1. Two has a strategy for $G_1(\mathcal{A}, \mathcal{C}) \implies$ Two has a strategy of the same level for $G_1(\mathcal{B}, \mathcal{D})$.
2. One does not have a strategy for $G_1(\mathcal{A}, \mathcal{C}) \implies$ One does not have a strategy at that same level for $G_1(\mathcal{B}, \mathcal{D})$.

- This relation is transitive.
- We say $G_1(\mathcal{A}, \mathcal{C}) \equiv G_1(\mathcal{B}, \mathcal{D})$ if $G_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_1(\mathcal{B}, \mathcal{D})$ and $G_1(\mathcal{B}, \mathcal{D}) \leq_{\text{II}} G_1(\mathcal{A}, \mathcal{C})$

# General Translation

If we can build the picture below, then $G_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_1(\mathcal{B}, \mathcal{D})$.




① $x \in \overleftarrow{T}_{\mathrm{I},n}(\mathcal{B}) \Rightarrow \overrightarrow{T}_{\mathrm{II},n}(x, \mathcal{B}) \in \mathcal{B}$

② $x_n \in \overleftarrow{T}_{\mathrm{I},n}(\mathcal{B}_n)$ and $\{x_n : n \in \omega\} \in \mathcal{E} \Rightarrow \{\overrightarrow{T}_{\mathrm{II},n}(x_n, \mathcal{B}_n) : n \in \omega\} \in \mathcal{D}$

# Dual Games

**Definition**

Define $G_1(\mathcal{A}, \mathcal{C})$ and $G_1(\mathcal{B}, \mathcal{D})$ are **dual** it

1. Two has a full information strategy for $G_1(\mathcal{A}, \mathcal{C})$ if and only if One has a full information strategy for $G_1(\mathcal{B}, \mathcal{D})$.
2. Two has a strategy that only looks at the turn number and most recent move in $G_1(\mathcal{A}, \mathcal{C})$ if and only if One has a strategy that only looks at the turn number in $G_1(\mathcal{B}, \mathcal{D})$.

The classic example is that the Rothberger game and the point-open game are dual.

Thanks to Clontz, we have a nice tool for checking if two selection games are dual.

# Different Cover Types

## $\omega$-Covers

A non-trivial open cover $\mathscr{U}$ of $X$ is an $\omega$-cover if for each finite $F \subseteq X$, there is a $U \in \mathscr{U}$ so that $F \subseteq U$. We use $\Omega(X)$ to refer to the collection of $\omega$-covers.

## $k$-Covers

A non-trivial open cover $\mathscr{U}$ of $X$ is an $k$-cover if for each compact $K \subseteq X$, there is a $U \in \mathscr{U}$ so that $K \subseteq U$. We use $\mathcal{K}(X)$ to refer to the collection of $k$-covers.

## $\gamma$-Cover Variants

- $\Gamma_\omega(X)$ is the countable $\omega$-covers so that every finite $F \subseteq X$ is covered by cofinitely many open sets.
- $\Gamma_k(X)$ is the countable $k$-covers so that every compact $K \subseteq X$ is covered by cofinitely many open sets.

# Notions Related to Clustering

Suppose $X$ is a topological space.

- $\Omega_{X,x} = \{A \subseteq X : x \in \overline{A}\}$.
- $\Gamma_{X,x} = \{s \in X^\omega : s \to x\}$.
- $\mathcal{D}_X = \{A \subseteq X : A \text{ is dense}\}$.
- $\mathrm{CD}(X) = \{A \subseteq X : A \text{ is closed and discrete}\}$.
- For $x \in X$, $\mathcal{N}_{X,x}$ is the set of open nhoods of $x$.
- If $A \subseteq X$ , then $\mathcal{N}_X(A) = \{U \in \mathcal{T}_X : A \subseteq U\}$.
- If $\mathcal{A} \subseteq \mathcal{P}(X)$, then $\mathcal{N}_X[\mathcal{A}] = \{N_X(A) : A \in \mathcal{A}\}$.

# The Games

We consider the following games in our result:

- $G_1(\Omega(X), \Omega(X))$, a variant of the Rothberger game.
- $G_1(\mathcal{N}_X[X^{<\omega}], \neg\Omega(X))$ and $G_1(\mathcal{N}_X[X^{<\omega}], \neg\Gamma_\omega(X))$, variants of the point-open game.
- $G_1(\Omega_{\mathrm{MC}_p(X),\mathbf{0}}, \Omega_{\mathrm{MC}_p(X),\mathbf{0}})$, the strong countable fan tightness game.
- $G_1(\mathcal{D}_{\mathrm{MC}_p(X)}, \Omega_{\mathrm{MC}_p(X),\mathbf{0}})$, the strong countable dense fan tightness game.
- $G_1(\mathcal{N}_{\mathrm{MC}_p(X),\mathbf{0}}, \neg\Omega_{\mathrm{MC}_p(X),\mathbf{0}})$ and $G_1(\mathcal{N}_{\mathrm{MC}_p(X),\mathbf{0}}, \neg\Gamma_{\mathrm{MC}_p(X),\mathbf{0}})$, versions of Gruenhage's games.
- $G_1(\mathcal{T}_{\mathrm{MC}_p(X)}, \mathrm{CD}_{\mathrm{MC}_p(X)})$, a game introduced by Tkachuk.

There are also the compact versions of all of these.

# The Big Theorem

**Theorem (Caruvana and Holshouser, 2023)**

Suppose $X$ is Hausdorff and completely regular. Then

- In group 1,

$$G_1(\Omega(X), \Omega(X)) \equiv G_1(\Omega_{\mathrm{MC}_p(X),\mathbf{0}}, \Omega_{\mathrm{MC}_p(X),\mathbf{0}}) \equiv G_1(\mathcal{D}_{\mathrm{MC}_p(X)}, \Omega_{\mathrm{MC}_p(X),\mathbf{0}})$$

- and in group 2,

$$\begin{aligned} G_1(\mathcal{N}_X[X^{<\omega}], \neg\Omega(X)) &\equiv G_1(\mathcal{N}_X[X^{<\omega}], \neg\Gamma_\omega(X)) \\ &\equiv G_1(\mathcal{N}_{\mathrm{MC}_p(X),\mathbf{0}}, \neg\Omega_{\mathrm{MC}_p(X),\mathbf{0}}) \\ &\equiv G_1(\mathcal{N}_{\mathrm{MC}_p(X),\mathbf{0}}, \neg\Gamma_{\mathrm{MC}_p(X),\mathbf{0}}) \\ &\equiv G_1(\mathcal{T}_{\mathrm{MC}_p(X)}, \mathrm{CD}_{\mathrm{MC}_p(X)}) \end{aligned}$$

- Group 1 and Group 2 are dual to each other.

The theorem is also true if finite is replaced with compact.

# Sample Proof Sketch

To prove that $G_1(\mathcal{N}_X[X^{<\omega}], \neg\Omega(X)) \leq_{\mathrm{II}} G_1(\mathcal{T}_{\mathrm{MC}_p(X)}, \mathrm{CD}_{\mathrm{MC}_p(X)})$, use

$$\overleftarrow{T}_{\mathrm{I},n}([\Phi; F, \varepsilon]) = \mathcal{N}_X(F)$$

and define $\overrightarrow{T}_{\mathrm{II},n}(U, [\Phi; F, \varepsilon])$ by first finding $V_{F,U}$ so that $F \subseteq V_{F,U} \subseteq \overline{V_{F,U}} \subseteq U$, then creating $f_{\Phi,F,U,n} : X \to \mathbb{R}$ by

$$f_{\Phi,F,U,n}(x) = \begin{cases} n & x \in \overline{X \setminus \overline{V_{F,U}}} \\ \max \Phi(x) & \text{otherwise} \end{cases}$$

and finally setting

$$\overrightarrow{T}_{\mathrm{II},n}(U, [\Phi; F, \varepsilon]) = \begin{cases} \check{f}_{\Phi,F,U,n} & F \subseteq U \\ \mathbf{0} & \text{otherwise} \end{cases}$$

# Some Corollaries

**Corollary (Caruvana and Holshouser, 2023)**

Suppose $X$ is Hausdorff and completely regular.

1. $X$ is countable if and only if $\mathrm{MC}_p(X)$ is metrizable.
2. $X$ is hemicompact if and only if $\mathrm{MC}_k(X)$ is metrizable.
3. $X$ is $\omega$-Rothberger if and only if $\mathrm{MC}_p(X)$ has strong countable fan tightness.
4. $X$ is $k$-Rothberger if and only if $\mathrm{MC}_k(X)$ has strong countable fan tightness.

# Thanks!

Thanks for Listening