Limited Information Strategies in Star Selection Games

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Selection Principles (Menger, Hurewicz 1924) (Scheepers 1996)

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are collections.

$S_1(\mathcal{A}, \mathcal{B})$

$S_1(\mathcal{A}, \mathcal{B})$ means that for all sequences A_n consisting of elements of $\mathcal{A}$, there are choices $x_n \in A_n$ so that $\{x_n : n \in \omega\} \in \mathcal{B}$.

$S_{\text{fin}}(\mathcal{A}, \mathcal{B})$

$S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ means that for all sequences A_n consisting of elements of $\mathcal{A}$, there are finite $F_n \subseteq A_n$ so that $\bigcup_n F_n \in \mathcal{B}$.

Let $\mathcal{O}(X)$ denote the open covers of X. A basic example of a selection principle is $S_1(\mathcal{O}(X), \mathcal{O}(X))$, a generalization of compactness that we refer to as **Rothberger**.

Selection Games

- We can view the selection principle $S_1(\mathcal{A}, \mathcal{B})$ as a game process wherein player I plays sets A_n and player II responds with $x_n \in A_n$.
- Player II wins if $\{x_n : n \in \omega\} \in \mathcal{B}$. We call this game $G_1(\mathcal{A}, \mathcal{B})$. Otherwise player I wins.
- In this game framework it's natural to impose information conditions on the players. These create a hierarchy of statements. In the Rothberger case, this looks like

$$X \text{ is ctbl } \rightarrow \text{ II wins } G_1(\mathcal{O}, \mathcal{O}) \rightarrow \text{ I doesn't win } G_1(\mathcal{O}, \mathcal{O}) \rightarrow S_1(\mathcal{O}, \mathcal{O})$$

Comparing Games

Definition

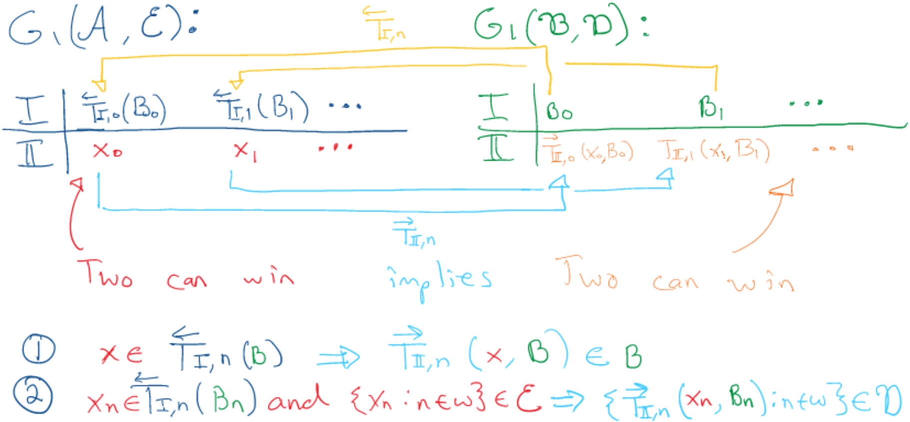
Define $G_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_1(\mathcal{B}, \mathcal{D})$ as the conjunction of the following implications.

1. Two has a strategy for $G_1(\mathcal{A}, \mathcal{C}) \implies$ Two has a strategy of the same level for $G_1(\mathcal{B}, \mathcal{D})$.
2. One does not have a strategy for $G_1(\mathcal{A}, \mathcal{C}) \implies$ One does not have a strategy at that same level for $G_1(\mathcal{B}, \mathcal{D})$.

- This relation is transitive.
- There is a fin version of all of this.

General Translation

If we can build the picture below, then $G_1(\mathcal{A}, \mathcal{C}) \leq_{\mathrm{II}} G_1(\mathcal{B}, \mathcal{D})$.



There is a small modification of this that works simultaneously for G_1 and G_{fin}.

ω-Covers

ω-Covers

A non-trivial open cover $\mathscr{U}$ of X is an ω-cover if for each finite $F \subseteq X$, there is a $U \in \mathscr{U}$ so that $F \subseteq U$. We use $\Omega(X)$ to refer to the collection of ω-covers.

Star Selections

Stars

If $\mathscr{U}$ is an open cover of X and $A \subseteq X$, then $\mathrm{St}(A, \mathscr{U}) = \bigcup\{U \in \mathscr{U} : U \cap A \neq \emptyset\}$.

Star Selections (Kocinac, 1999)

- The symbol $S_1^*(\mathcal{O}, \mathcal{B})$ says that for each sequence of open covers $\mathscr{U}_n$, there are open sets $U_n \in \mathscr{U}_n$ so that $\{\mathrm{St}(U_n, \mathscr{U}_n) : n \in \omega\} \in \mathcal{B}$.
- The symbol $SS_{\mathcal{A}}^*(\mathcal{O}, \mathcal{B})$ says for each sequence of open covers $\mathscr{U}_n$, there is a sequence of $A_n \in \mathcal{A}$ so that $\{\mathrm{St}(A_n, \mathscr{U}_n) : n \in \omega\} \in \mathcal{B}$.

Star covering properties appeared at least as early as 1991 (E.K. van Douwen, G.M. Reed, A.W. Roscoe and I.J. Tree).

Star Selection Principles are Selection Principles

Constellations and Galaxies

- If $\mathscr{U}$ is an open cover of X and $\mathcal{A}$ is a collection of subsets of X, then $\mathrm{Cons}(\mathcal{A}, \mathscr{U}) = \{\mathrm{St}(A, \mathscr{U}) : A \in \mathcal{A}\}$.
- If $\mathscr{C}$ is a collection of open covers of X and $f : \mathscr{C} \to \mathcal{P}(\mathcal{P}(X))$, then $\mathrm{Gal}(f, \mathscr{C}) = \{\mathrm{Cons}(f(\mathscr{U}), \mathscr{U}) : \mathscr{U} \in \mathscr{C}\}$.

Star Selections

- $S_1^*(\mathcal{O}, \mathcal{B})$ is equivalent to $S_1(\mathrm{Gal}(\mathrm{id}, \mathcal{O}), \mathcal{B})$.
- $SS_{\mathcal{A}}^*(\mathcal{O}, \mathcal{B})$ is equivalent to $S_1(\mathrm{Gal}(\mathcal{A}, \mathcal{O}), \mathcal{B})$.

With this equivalence in mind, we will reference the corresponding games $G_1^*(\mathcal{O}, \mathcal{B})$ and $SG_1^*(\mathcal{O}, \mathcal{B})$ and note that the translation theorem can be applied to it.

The Pixley-Roy Hyperspace

Definition (Pixley and Roy, 1969)

We define the topological space $\mathrm{PR}(X)$ as follows:

- points in $\mathrm{PR}(X)$ are finite subsets of X, and
- A basic open set has the from $[F, U] = \{G \subseteq X : F \subseteq G \subseteq U\}$, where F and G are finite and U is open in X.

This topology is finer than the Fell topology and was initially created as an interesting space for counter-examples.

The Star Selection Game on the Pixley-Roy Hyperspace

Theorem (Sakai, 2014)

Suppose X is regular. Then
$S^*_\square(\mathcal{O}(\mathrm{PR}(X)), \mathcal{O}(\mathrm{PR}(X))) \implies S_\square(\Omega(X), \Omega(X))$.

We boost this up to the following result.

Theorem (Caruvana and Holshouser, 2022)

Assume X is regular. Then
$G^*_\square(\mathcal{O}(\mathrm{PR}(X)), \mathcal{O}(\mathrm{PR}(X))) \leq_{\mathrm{II}} G_\square(\Omega(X), \Omega(X))$.

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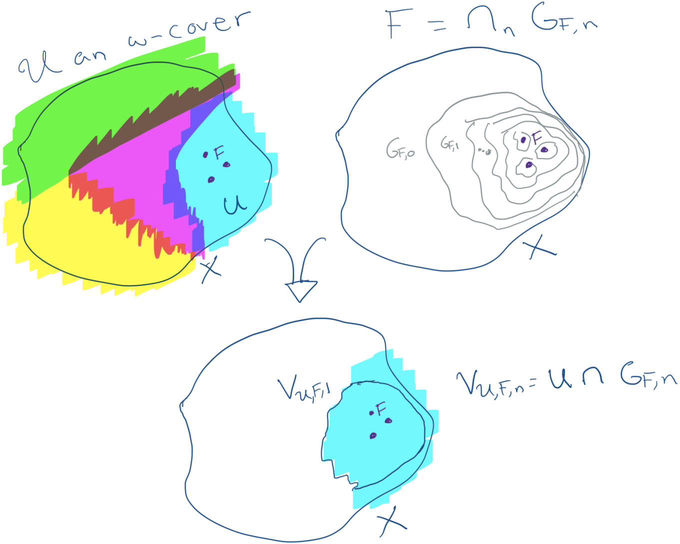
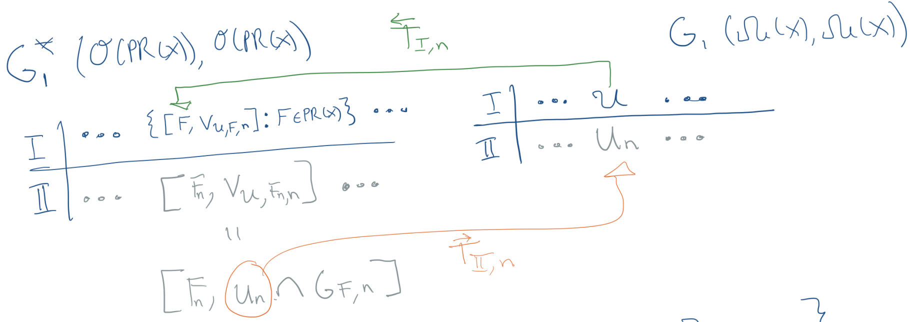


$G^*_\square(\mathcal{O}(\mathrm{PR}(X)), \mathcal{O}(\mathrm{PR}(X))) \leq_{\mathrm{II}} G_\square(\Omega(X), \Omega(X))$



Check that when $\{[F_n, V_{\mathcal{U},F_n,n}] : n \in \omega\}$ cover $PR(X)$, the $\{U_n : n \in \omega\}$ are an ω-cover of X.

A Useful Refinement

Theorem (Caruvana and Holshouser 2022)

There is a version of the translation theorem where $\overleftarrow{T}_{\mathrm{I},n}$ doesn't have to pick out an individual from $\mathcal{A}$, but instead it picks out a subset of $\mathcal{A}$.

Star Selection in Uniform Spaces

Definition

Given a uniform space $(X, \mathcal{E})$ and a collection $\mathscr{U}$ of subsets of X, $\mathscr{U}$ is a **uniform cover** of X (with respect to $\mathcal{E}$) if there exists $E \in \mathcal{E}$ so that $\{E[x] : x \in X\}$ is a refinement of $\mathscr{U}$.

We will say a uniform cover is an **open uniform cover** if it consists of open sets.

Let $\mathcal{C}_{\mathcal{E}}(X)$ be the collection of all open uniform covers with respect to $\mathcal{E}$.

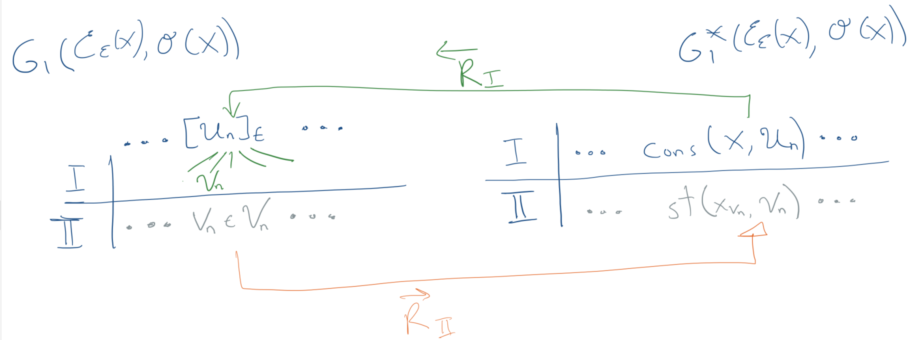
Theorem (Caruvana and Holshouser, 2022)

Let $(X, \mathcal{E})$ be a uniform space. Then

$$\mathsf{G}_{\square}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X) \equiv \mathsf{SG}^*_{X,\square}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X) \equiv \mathsf{G}^*_{\square}(\mathcal{C}_{\mathcal{E}}(X), \mathcal{O}_X).$$

This theorem extends a result of Kocinac (2003).

$\mathsf{G}_\square(\mathcal{C}_\mathcal{E}(X), \mathcal{O}_X) \leq_{\text{II}} \mathsf{SG}^*_{X,\square}(\mathcal{C}_\mathcal{E}(X), \mathcal{O}_X)$



- $\mathscr{U} \sim_E \mathscr{V}$ means $\mathrm{Cons}(X, \mathscr{U}) = \mathrm{Cons}(X, \mathscr{V})$.
- For each open $V \in \mathscr{T}_X$, choose $x_V \in V$.
- Check that if $V_n \in \mathscr{V}_n \in [\mathscr{U}_n]_E$, then $\mathrm{St}(x_{V_n}, \mathscr{V}_n) \in \mathrm{Cons}(X, \mathscr{U}_n)$.
- Check that if $X = \bigcup_n V_n$, then $X = \bigcup_n \mathrm{St}(x_{V_n}, \mathscr{V}_n)$.

Thanks!

Thanks for Listening