Translation for a Star Selection Game on Pixley-Roy

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Selection Principles (Menger, Hurewicz 1924) (Scheepers 1996)

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are collections.

$S_1(\mathcal{A}, \mathcal{B})$

$S_1(\mathcal{A}, \mathcal{B})$ means that for all sequences A_n consisting of elements of $\mathcal{A}$, there are choices $x_n \in A_n$ so that $\{x_n : n \in \omega\} \in \mathcal{B}$.

$S_{\text{fin}}(\mathcal{A}, \mathcal{B})$

$S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ means that for all sequences A_n consisting of elements of $\mathcal{A}$, there are finite $F_n \subseteq A_n$ so that $\bigcup_n F_n \in \mathcal{B}$.

Let $\mathcal{O}(X)$ denote the open covers of X. A basic example of a selection principle is $S_1(\mathcal{O}(X), \mathcal{O}(X))$, a generalization of compactness that we refer to as Rothberger.

Selection Games

- We can view the selection principle $S_1(\mathcal{A}, \mathcal{B})$ as a game process wherein player I plays sets A_n and player II responds with $x_n \in A_n$.
- Player II wins if $\{x_n : n \in \omega\} \in \mathcal{B}$. We call this game $G_1(\mathcal{A}, \mathcal{B})$. Otherwise player I wins.
- In this game framework it's natural to impose information conditions on the players. These create a hierarchy of statements. In the Rothberger case, this looks like

$$X \text{ is ctbl } \rightarrow \text{ II wins } G_1(\mathcal{O}, \mathcal{O}) \rightarrow \text{ I doesn't win } G_1(\mathcal{O}, \mathcal{O}) \rightarrow S_1(\mathcal{O}, \mathcal{O})$$

Game Equivalence

Definition

Define $G_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_1(\mathcal{B}, \mathcal{D})$ as the conjunction of the following implications.

1. Two has a limited information strategy for $G_1(\mathcal{A}, \mathcal{C}) \implies$ Two has a strategy of the same level for $G_1(\mathcal{B}, \mathcal{D})$
2. One does not have a limited information strategy for $G_1(\mathcal{A}, \mathcal{C}) \implies$ One does not have a strategy at that same level for $G_1(\mathcal{B}, \mathcal{D})$

- This relation is transitive.
- There is a fin version of all of this.

General Translation

Theorem (Caruvana and Holshouser 2019)

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be collections. Suppose there are functions

- $\overleftarrow{T}_{\mathrm{I},n} : \mathcal{B} \to \mathcal{A}$ and
- $\overrightarrow{T}_{\mathrm{II},n} : \bigcup \mathcal{A} \times \mathcal{B} \to \bigcup \mathcal{B}$

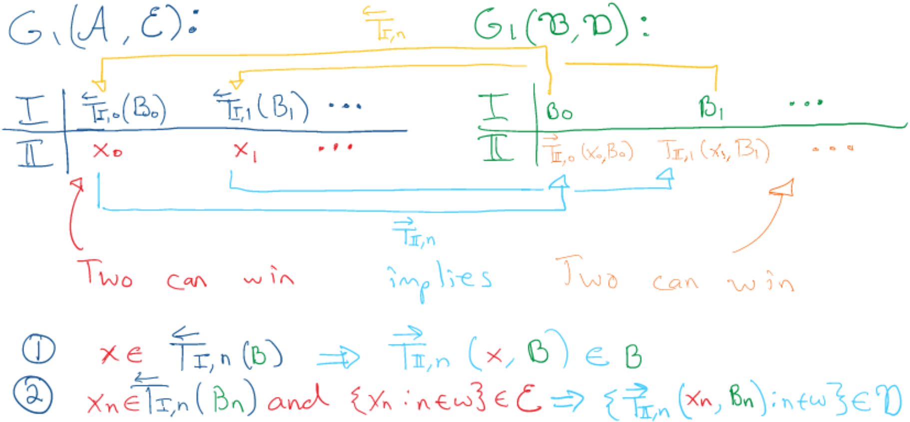
so that

1. if $x \in \overleftarrow{T}_{\mathrm{I},n}(B)$, then $\overrightarrow{T}_{\mathrm{II},n}(x, B) \in B$ and
2. if $x_n \in \overleftarrow{T}_{\mathrm{I},n}(B_n)$ for all n, then

$$\{x_n : n \in \omega\} \in \mathcal{C} \implies \left\{\overrightarrow{T}_{\mathrm{II},n}(x_n, B_n) : n \in \omega\right\} \in \mathcal{D}$$

Then $G_1(\mathcal{A}, \mathcal{C}) \leq_{\mathrm{II}} G_1(\mathcal{B}, \mathcal{D})$.

General Translation



There is a small modification of this that works simultaneously for G_1 and G_{fin}.

ω-Covers

ω-Covers

A non-trivial open cover $\mathscr{U}$ of X is an ω-cover if for each finite $F \subseteq X$, there is a $U \in \mathscr{U}$ so that $F \subseteq U$. We use $\Omega(X)$ to refer to the collection of ω-covers.

Star Selections

Stars

If $\mathscr{U}$ is an open cover of X and $A \subseteq X$, then
$\mathrm{St}(A, \mathscr{U}) = \bigcup\{U \in \mathscr{U} : U \cap A \neq \emptyset\}$.

Star Selections (Kocinac, 1999)

- The symbol $S_1^*(\mathcal{O}, \mathcal{B})$ says that for each sequence of open covers $\mathscr{U}_n$, there are open sets $U_n \in \mathscr{U}_n$ so that $\{\mathrm{St}(U_n, \mathscr{U}_n) : n \in \omega\} \in \mathcal{B}$.
- The symbol $SS_{\mathcal{A}}^*(\mathcal{O}, \mathcal{B})$ says for each sequence of open covers $\mathscr{U}_n$, there is a sequence of $A_n \in \mathcal{A}$ so that $\{\mathrm{St}(A_n, \mathscr{U}_n) : n \in \omega\} \in \mathcal{B}$.

Star covering properties appeared at least as early as 1991 (E.K. van Douwen, G.M. Reed, A.W. Roscoe and I.J. Tree).

Star Selection Principles are Selection Principles

Constellations and Galaxies

- If $\mathscr{U}$ is an open cover of X and $\mathcal{A}$ is a collection of subsets of X, then $\mathrm{Cons}(\mathcal{A}, \mathscr{U}) = \{\mathrm{St}(A, \mathscr{U}) : A \in \mathcal{A}\}$.
- If $\mathscr{C}$ is a collection of open covers of X and $f : \mathscr{C} \to \mathcal{P}(\mathcal{P}(X))$, then $\mathrm{Gal}(f, \mathscr{C}) = \{\mathrm{Cons}(f(\mathscr{U}), \mathscr{U}) : \mathscr{U} \in \mathscr{C}\}$.

Star Selections

- $S_1^*(\mathcal{O}, \mathcal{B})$ is equivalent to $S_1(\mathrm{Gal}(\mathrm{id}, \mathcal{O}), \mathcal{B})$.
- $SS_{\mathcal{A}}^*(\mathcal{O}, \mathcal{B})$ is equivalent to $S_1(\mathrm{Gal}(\mathcal{A}, \mathcal{O}), \mathcal{B})$.

With this equivalence in mind, we will reference the corresponding game $G_1^*(\mathcal{O}, \mathcal{B})$ and note that the translation theorem can be applied to it.

The Pixley-Roy Hyperspace

Definition (Pixley and Roy, 1969)

If $\mathcal{A}$ is an ideal of compact subsets of X, we define the topological space $\mathrm{PR}_{\mathcal{A}}(X)$ as follows:

- The space is $\mathcal{A}$,
- A basic open set has the from $[A, U] = \{B \in \mathcal{A} : A \subseteq B \subseteq U\}$.

$\mathrm{PR}(X)$ is the special case when $\mathcal{A}$ is the finite subsets of X.

Star Selection on Pixley-Roy

Theorem (Sakai, 2014)

Suppose X is regular. Then $S^*_\Box(\mathcal{O}(\mathrm{PR}(X)), \mathcal{O}(\mathrm{PR}(X)))$ implies $S_\Box(\Omega(X), \Omega(X))$.

- Can we get this to work for limited information strategies?
- Can we get this to work for longer games?
- Can we get this to work for $\mathrm{PR}_{\mathcal{A}}(X)$?

Some Topological Results

Theorem

If X is regular and $\mathcal{A}$ is an ideal of compact subsets of X, then

$$\psi(X) \leq \psi_k(X) \leq L_{\mathrm{st}}(\mathrm{PR}_{\mathcal{A}}(X))$$

Where

- $\psi(X)$ is the pseudocharacter of X: the least κ so that every point is the intersection κ open sets,
- $\psi_k(X)$ is the k-pseudocharacter of X: the least κ so that every compact set is the intersection κ open sets, and
- $L_{\mathrm{st}}(Y)$ is the star Lindelof degree of Y: the least κ so that every open cover of X has a κ-sized subcollection whose stars form a cover.

Corollaries and Combinatorics

Topological Corollary

Assume $\mathrm{PR}(X)$ is star Lindelof and X is regular. Then X is a G_δ space. In fact, each compact set is G_δ.

Combinatorial Proposition

Suppose X satisfies $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ and fix a bijection $\beta : \omega^2 \to \omega$. Then, for a sequence $\langle A_n : n \in \omega \rangle$ from $\mathcal{A}$, there is a selection $x_n \in A_n$ so that $\langle x_{\beta(n,m)} : m \in \omega \rangle \in \mathcal{B}$ for all $n \in \omega$.

The Star Selection Game on the Pixley-Roy Hyperspace

Theorem (Caruvana and Holshouser, 2022)

Assume X is regular. Then
$G^*_\square(\mathcal{O}(\mathrm{PR}(X)), \mathcal{O}(\mathrm{PR}(X))) \leq_{\mathrm{II}} G_\square(\Omega(X), \Omega(X))$.

- WLOG assume $\mathrm{PR}(X)$ is star Lindelof. Then X is a G_δ space, say $\{x\} = \bigcap_n G_{x,n}$. For $F \subseteq X$ finite, set $G_{F,n} = \bigcup_{x \in F} G_{x,n}$. We assume the $G_{F,n}$ are descending in n.
- Choose a function $\gamma : \Omega(X) \times \mathrm{PR}(X) \to \mathcal{T}_X$ so that $\gamma(\mathscr{U}, F) \in \mathscr{U}$ and $F \subseteq \gamma(\mathscr{U}, F)$. Then set

$$\mathscr{V}_{\mathscr{U},n} = \{[F, \gamma(\mathscr{U}, F) \cap G_{F,n}] : F \in \mathrm{PR}(X)\}.$$

Proof Outline

- Define $\overleftarrow{T}_{\mathrm{I},n}(\mathscr{U}) = \mathrm{Cons}(\mathscr{V}_{\mathscr{U},n}, \mathscr{V}_{\mathscr{U},n})$.
- Choose a function $\chi_{\mathscr{U},n} : \mathrm{Cons}(\mathscr{V}_{\mathscr{U},n}, \mathscr{V}_{\mathscr{U},n}) \to \mathrm{PR(X)}$ so that, letting $F = \chi_{\mathscr{U},n}(W)$,

$$W = \mathrm{St}([F, \gamma(\mathscr{U}, F) \cap G_{F,n}], \mathscr{V}_{\mathscr{U},n}).$$

- Define $\overrightarrow{T}_{\mathrm{II},n}(W, \mathscr{U}) = \gamma(\mathscr{U}, \chi_{\mathscr{U},n}(W))$.
- Check the two criteria for $\overleftarrow{T}_{\mathrm{I},n}$ and $\overrightarrow{T}_{\mathrm{II},n}$.
  - Property 1 is by design,
  - Property 2 uses the combinatorial proposition, the fact that the $G_{F,n}$ are descending, and the fact that the F are finite.

Thanks!

Thanks for Listening