Selection Games in Hyperspace Topologies

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Outline

1. Thank the Organizers
2. Hyperspace and Selection Game Preliminaries
3. Prior Work of Kočinac et al. and Li
4. Generalizing and Unifying Results

Selection Principles (Menger, Hurewicz 1924) (Scheepers 1996)

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are collections.

$S_1(\mathcal{A}, \mathcal{B})$

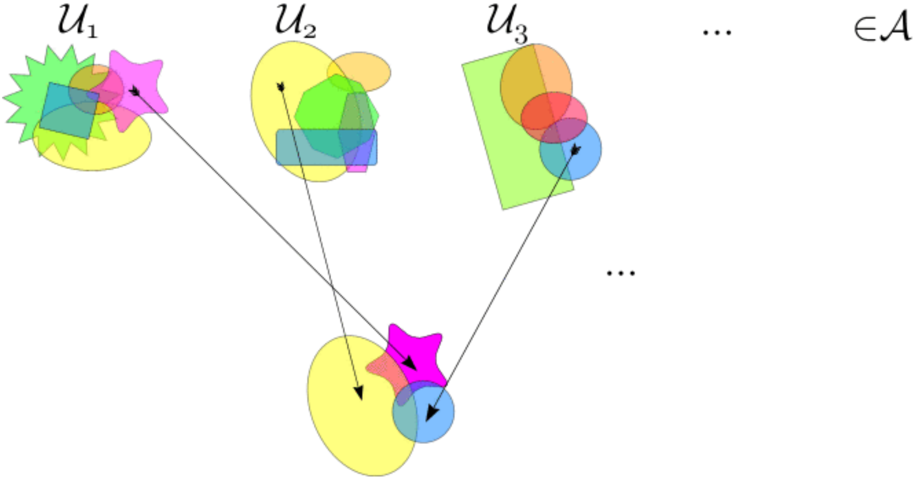
$S_1(\mathcal{A}, \mathcal{B})$ means that for all sequences A_n consisting of elements of $\mathcal{A}$, there are choices $x_n \in A_n$ so that $\{x_n : n \in \omega\} \in \mathcal{B}$.

$S_{\text{fin}}(\mathcal{A}, \mathcal{B})$

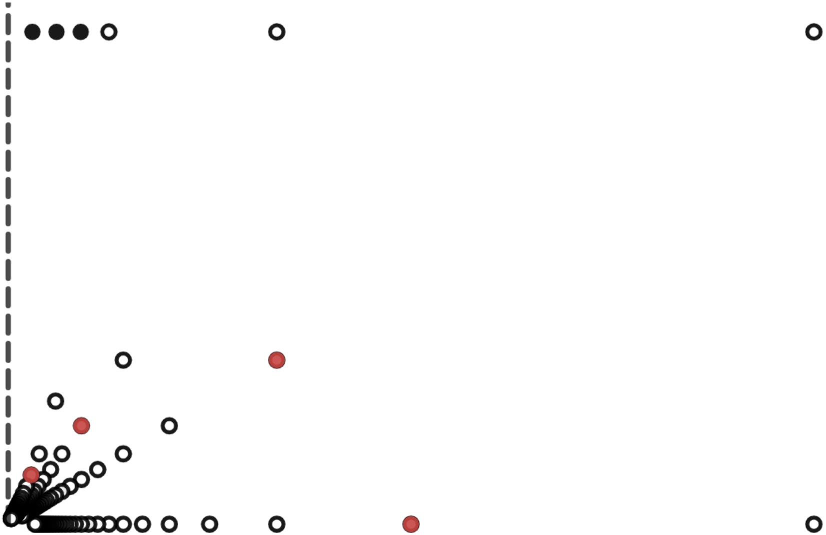
$S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ means that for all sequences A_n consisting of elements of $\mathcal{A}$, there are finite $F_n \subseteq A_n$ so that $\bigcup_n F_n \in \mathcal{B}$.

- Let $\mathcal{O}(X)$ denote the open covers of X.
- Let $\mathscr{D}_X$ denote the dense subsets of X.
- Let $\Omega_{X,x}$ denote the sets $A \subseteq X$ such that $x \in \overline{A}$.

Rothberger $S_1(\mathcal{O}(X), \mathcal{O}(X))$



Strong Countable Fan Tightness $S_1(\Omega_{X,x}, \Omega_{X,x})$



Strategies

- A **perfect information (PI) strategy** for either player One or Two is a strategy for responding to the other player that takes as inputs all of the previous plays of the game.
- A **Markov strategy** for player Two is a strategy that takes as inputs the current turn number and the most recent play of player One.
- A **pre-determined (PD) strategy** for player One is a strategy where the only input is the current turn number.
- A strategy is **winning** if following the strategy guarantees that the player will win the game.

Strategies

- Playing according to a PI strategy for One:

I	$\sigma(\emptyset)$	$\sigma(x_0)$	$\sigma(x_0, x_1)$	$\cdots$
II	x_0	x_1	x_2	$\cdots$

- Playing according to a PI strategy for Two:

I	A_0	A_1	A_2	$\cdots$
II	$\tau(A_0)$	$\tau(A_0, A_1)$	$\tau(A_0, A_1, A_2)$	$\cdots$

- Playing according to a Markov strategy for Two:

I	A_0	A_1	A_2	$\cdots$
II	$\tau(A_0, 0)$	$\tau(A_1, 1)$	$\tau(A_2, 2)$	$\cdots$

- Playing according to a PD strategy for One:

I	$\sigma(0)$	$\sigma(1)$	$\sigma(2)$	$\cdots$
II	x_0	x_1	x_2	$\cdots$

A Strength Hierarchy

For the selection game $G_1(\mathcal{A}, \mathcal{B})$:

$$\begin{array}{rcl} \text{Two has a winning Markov Strategy} & & \\ \Downarrow & & \\ \text{Two has a winning PI strategy} & & \\ \Downarrow & & \\ \text{One has no winning PI strategy} & & \\ \Downarrow & & \\ \text{One has no winning PD strategy} & \iff & S_1(\mathcal{A}, \mathcal{B}) \end{array}$$

Game Equivalence

Definition

Define $G_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_1(\mathcal{B}, \mathcal{D})$ as the conjunction of the following implications.

1. Two has Mark in $G_1(\mathcal{A}, \mathcal{C}) \implies$ Two has a Mark in $G_1(\mathcal{B}, \mathcal{D})$
2. Two has PI strat in $G_1(\mathcal{A}, \mathcal{C}) \implies$ Two has a PI strat in $G_1(\mathcal{B}, \mathcal{D})$
3. One has no PI strat in $G_1(\mathcal{A}, \mathcal{C}) \implies$ One has no PI strat in $G_1(\mathcal{B}, \mathcal{D})$
4. One has no PD strat in $G_1(\mathcal{A}, \mathcal{C}) \implies$ One has no PD strat in $G_1(\mathcal{B}, \mathcal{D})$

- This relation is transitive.
- if $G_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_1(\mathcal{B}, \mathcal{D})$ and $G_1(\mathcal{B}, \mathcal{D}) \leq_{\text{II}} G_1(\mathcal{A}, \mathcal{C})$, we say the two games are **equivalent** and write $G_1(\mathcal{A}, \mathcal{C}) \equiv G_1(\mathcal{B}, \mathcal{D})$.
- There is a fin version of all of this.

The Upper Fell Hyperspace Topology

Let $\mathbb{F}(X)$ denote the collection of closed subsets of X.

The Upper Fell Topology (Fell 1962)

The **Upper Fell Topology** on $\mathbb{F}(X)$ is generated by basic open sets of the form

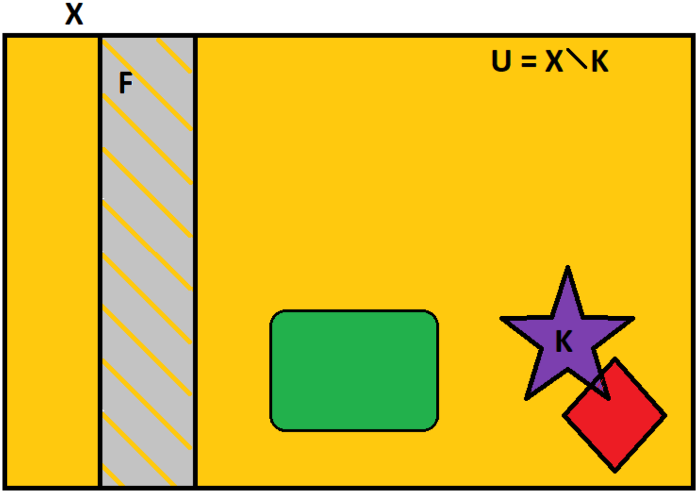
$$(X \setminus K)^+ := \{F \in \mathbb{F}(X) : F \subseteq (X \setminus K)\}$$

where K is a compact subset of X.

Let $\mathbb{F}^+(X)$ denote $\mathbb{F}(X)$ endowed with the upper Fell topology.
Note that this can also be done with finite subsets of X in place of compact sets.

The Upper Fell Hyperspace Topology

F with an open neighborhood $(X \setminus K)^+$.



The Upper Fell Hyperspace Topology

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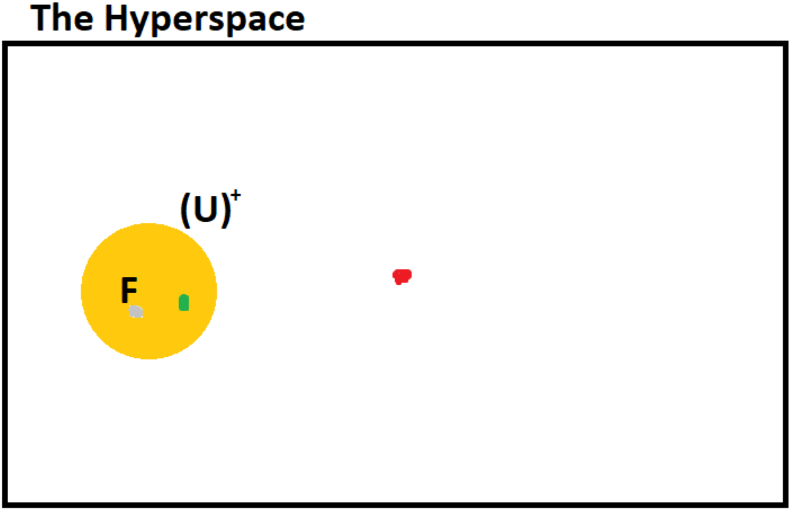

The Fell Hyperspace Topology

The Fell Topology (Fell 1962)

The **Fell Topology** on $\mathbb{F}(X)$ is generated by basic open sets of the form

$$[K; V_1, \cdots, V_n] := \{F \in \mathbb{F}(X) : F \subseteq (X \setminus K) \text{ and } F \cap V_j \neq \emptyset\}$$

where $K \subseteq X$ is compact and the $V_1, \cdots, V_n$ are open subsets of X.

Let $\mathbb{F}(X)$ denote $\mathbb{F}(X)$ endowed with the upper Fell topology. Note that this can also be done with finite subsets of X in place of compact sets.

Di Maio, Kočinac, and Meccariello

A non-trivial open cover $\mathscr{U}$ of X is a k-**cover** if for all compact $K \subseteq X$, there is a $U \in \mathscr{U}$ so that $K \subseteq U$. Let $\mathcal{K}(X)$ denote the collection of k-covers of X.

Theorem 2005

- $S_\square(\mathcal{K}(X), \mathcal{K}(X))$ if and only if $S_\square(\mathscr{D}_{\mathbb{F}^+(X)}, \mathscr{D}_{\mathbb{F}^+(X)})$.
- If $G \subseteq X$ is closed, then $S_\square(\mathcal{K}(X \setminus G), \mathcal{K}(X \setminus G))$ is equivalent to $S_\square(\Omega_{\mathbb{F}^+(X),G}, \Omega_{\mathbb{F}^+(X),G})$.
- The Hurewicz versions of the selection principles are also equivalent.
- All of the above is true if compact sets are replaced with finite sets and k-covers are replaced with ω-covers.

Li

A non-trivial open cover $\mathscr{U}$ of X is a k_F**-cover** if for all compact $K \subseteq X$, and all finite sequences of open sets $V_1, \cdots, V_n \subseteq X$ with the property that $(X \setminus K) \cap V_j \neq \emptyset$, there is a $U \in \mathscr{U}$ and a finite $F \subseteq X$ so that

- $K \subseteq U$,
- $F \cap V_j \neq \emptyset$ for all j, and
- $F \cap U = \emptyset$.

$\mathcal{K}_F(X)$ is the collection of k_F-covers of X.

Theorem 2016

All of what Kocinac et al. proved in 2005 holds when $\mathcal{K}$ is replaced with $\mathcal{K}_F$ and $\mathbb{F}^+(X)$ is replaced with $\mathbb{F}(X)$.

Generalizing and Unifying

To expand on these results, we did the following.

1. We worked with arbitrary ideals instead of compact/finite sets.
2. We worked with selection games and different strength strategies instead of selection principles.
3. We came up with a way to write all these proofs without working each individual situation separately.

Idealized Definitions

Let $\mathcal{A}$ be a proper ideal consisting of closed subsets of X.

- The **upper $\mathcal{A}$-topology** on $\mathbb{F}(X)$ is generated by basic open sets of the form
$$(X \setminus A)^+ = \{F \in \mathbb{F}(X) : F \cap A = \emptyset\}$$
where $A \in \mathcal{A}$. Denote this as $\mathbb{F}(X, \mathcal{A}^+)$.
- The **$\mathcal{A}$-topology** on $\mathbb{F}(X)$ is generated by basic open sets of the form
$$[A; V_1, \cdots, V_n] = \{F \in \mathbb{F}(X) : F \cap A = \emptyset \text{ and } F \cap V_j \neq \emptyset\}$$
where $A \in \mathcal{A}$. Denote this as $\mathbb{F}(X, \mathcal{A})$.
- A non-trivial open cover $\mathscr{U}$ of X is an **$\mathcal{A}$-cover** if for all $A \in \mathcal{A}$, there is a $U \in \mathscr{U}$ so that $A \subseteq U$. Denote these as $\mathcal{O}(X, \mathcal{A})$.
- Similarly, we can define **$\mathcal{A}_F$-covers** and denote the set of $\mathcal{A}_F$ covers as $\mathcal{O}_F(X, \mathcal{A})$.

A Helpful Translation Theorem

Carurvana, Holshouser 2020

Assume that $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are collections and that $\bigcup \mathcal{C} \subseteq \bigcup \mathcal{A}$ and $\bigcup \mathcal{D} \subseteq \bigcup \mathcal{B}$. Suppose that there is a bijection $\beta : \bigcup \mathcal{A} \to \bigcup \mathcal{B}$ so that

- $A \in \mathcal{A}$ if and only if $\beta[A] \in \mathcal{B}$, and
- $C \in \mathcal{C}$ if and only if $\beta[C] \in \mathcal{D}$.

Then $G_{\square}(\mathcal{A}, \mathcal{C}) \equiv G_{\square}(\mathcal{B}, \mathcal{D})$.

Applying the Translation Theorem

To translate from cover games on X to density/blade games on $\mathbb{F}(X)$, we use $\beta : \mathscr{T}_X \to \mathbb{F}(X)$ defined by $\beta(U) = X \setminus U$. This β has the following properties.

1. $\mathscr{U} \in \mathcal{O}(X, \mathcal{A})$ iff $\beta[\mathscr{U}] \in \mathcal{D}_{\mathbb{F}(X,\mathcal{A}^+)}$
2. β takes localized $\mathcal{A}$-covers to blades in $\mathbb{F}(X, \mathcal{A}^+)$.
3. $\mathscr{U} \in \mathcal{O}_F(X, \mathcal{A})$ iff $\beta[\mathscr{U}] \in \mathcal{D}_{\mathbb{F}(X,\mathcal{A})}$
4. β takes localized $\mathcal{A}_F$-covers to blades in $\mathbb{F}(X, \mathcal{A})$.
5. β respects groupability (and so will be useful for showing that the Hurewicz games are equivalent)

Caruvana and Holshouser

Theorem 2020

Fix a topological space X, $G \in \mathbb{F}(X)$, and ideals $\mathcal{A}$ and $\mathcal{B}$ consisting of closed sets. Then

1. $G_\square(\mathcal{O}(X, \mathcal{A}), \mathcal{O}(X, \mathcal{B})) \equiv G_\square(\mathcal{D}_{\mathbb{F}(X, \mathcal{A}^+)}, \mathcal{D}_{\mathbb{F}(X, \mathcal{B}^+)})$,
2. $G_\square(\mathcal{O}(X, X \setminus G, \mathcal{A}), \mathcal{O}(X, X \setminus G, \mathcal{B})) \equiv G_\square(\Omega_{\mathbb{F}(X, \mathcal{A}^+), G}, \Omega_{\mathbb{F}(X, \mathcal{B}^+), G})$,
3. $G_\square(\mathcal{O}(X, \mathcal{A}), \mathcal{O}^{gp}(X, \mathcal{B})) \equiv G_\square(\mathcal{D}_{\mathbb{F}(X, \mathcal{A}^+)}, \mathcal{D}^{gp}_{\mathbb{F}(X, \mathcal{B}^+)})$,

and

1. $G_\square(\mathcal{O}_F(X, \mathcal{A}), \mathcal{O}_F(X, \mathcal{B})) \equiv G_\square(\mathcal{D}_{\mathbb{F}(X, \mathcal{A})}, \mathcal{D}_{\mathbb{F}(X, \mathcal{B})})$,
2. $G_\square(\mathcal{O}_F(X, X \setminus G, \mathcal{A}), \mathcal{O}_F(X, X \setminus G, \mathcal{B})) \equiv G_\square(\Omega_{\mathbb{F}(X, \mathcal{A}), G}, \Omega_{\mathbb{F}(X, \mathcal{B}), G})$,
3. $G_\square(\mathcal{O}_F(X, \mathcal{A}), \mathcal{O}_F^{gp}(X, \mathcal{B})) \equiv G_\square(\mathcal{D}_{\mathbb{F}(X, \mathcal{A})}, \mathcal{D}^{gp}_{\mathbb{F}(X, \mathcal{B})})$.

Further Questions

- What about Pixley-Roy?
- In our paper, we applied β individually to the localized cover situations and to the Hurewicz context. Is there are a way to use β once and achieve all the results at once?
- Kočinac et al. and Li both draw connections between tightness on $\mathbb{F}(X)$ and covering properties of X. We can handle some of those connections within the selection games framework, but T-tightness eluded us. Can the T-tightness connection be made to fit into our framework?

Thanks!

Thanks for Listening