# Calibrating the Size of Complicated Quotients

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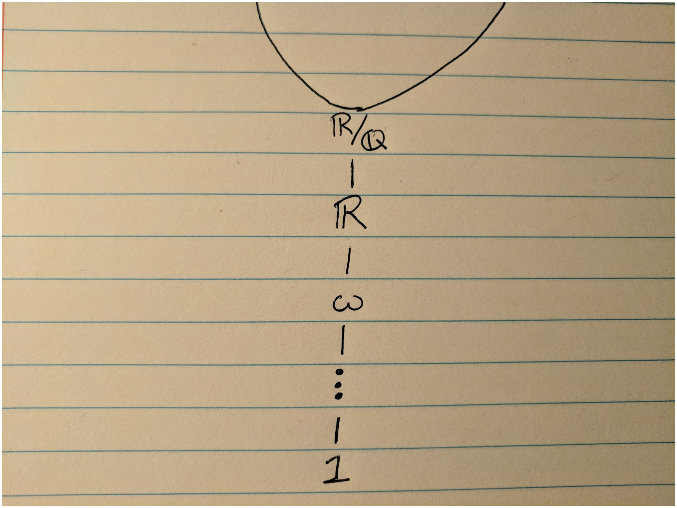
# The Big Picture

For $E$ a Borel equivalence relation, we have the following picture.



# The Trichotomy

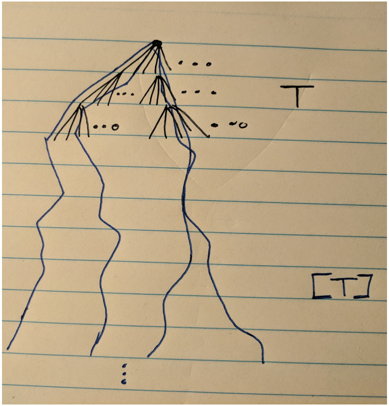
**J.Silver(1980) and L.Harrington-A.Kechris-A.Louveau(1990)**

- (Silver) If $E$ is a co-analytic equivalence relation on $\mathbb{R}$, then either $\mathbb{R}/E$ is countable or there is a continuous map $\varphi : \mathbb{R} \to \mathbb{R}$ so that if $\varphi(x) \neq \varphi(y)$, then $x$ and $y$ are $E$-inequivalent. In other words, $\mathbb{R}$ continuously embeds into $\mathbb{R}/E$.
- (Harrington-Kechris-Louveau) If $E$ is a Borel equivalence relation, then either $\mathbb{R}/E$ embeds into $\mathbb{R}$ or $\mathbb{R}/\mathbb{Q}$ embeds into $\mathbb{R}/E$ (in the same sense as the first bullet point).

The proof uses $\Sigma^1_1$ to classify the quotients.

# Combinatorics of $\mathbb{R}$

In descriptive set theory, $\mathbb{R}$ is replaced with $\omega^\omega$.
Closed sets are formed from the branches of trees on $\omega^n$.




Analytic sets are formed from the projection of the closed sets.
$\Sigma^1_1$ is those analytic sets generated from computable trees.

# Suslin Sets

- $A \subseteq \mathbb{R}^n$ is $\kappa$-**Suslin** if there is a tree $T$ on $\omega^n \times \kappa$ so that $A = p[T]$ where

$$p[T] = \{\vec{x} \in \mathbb{R}^n : (\exists f \in \kappa^\omega)(\forall k)[(x_0, \cdots, x_{n-1}, f)|_k \in T]\}.$$

- $A$ is co-$\kappa$-Suslin if $\mathbb{R}^n \setminus A$ is $\kappa$-Suslin.
- All analytic sets are $\omega$-Suslin.
- Using the axiom of choice, all sets of reals are $2^{\aleph_0}$-Suslin.
- This definition is a form of definability when a set is $\kappa$-Suslin for $\kappa < 2^{\aleph_0}$.

# The Harrington-Shelah Theorem

**L.Harrington-S.Shelah(1980)**

If $E$ is a co-$\kappa$-Suslin equivalence on $\mathbb{R}$ and $E$ is still an equivalence relation after Cohen-forcing, then either $\mathbb{R}/E$ embeds into $\kappa$ or $\mathbb{R}$ embeds into $\mathbb{R}/E$.

This proof uses infintary logic as its main tool. A collection of statements controlled by $L[T]$ replaces $\Sigma^1_1$.

- $E$ is still an equivalence relation after Cohen-forcing if $E$ is inside of a universe ZF plus the axiom of determinacy.
- Universes that satisfy the axiom of choice and have large cardinals contain universes of determinacy
- Suslin sets naturally occur in those universes of determinacy.

# The Harrington-Shelah Theorem

**L.Harrington-S.Shelah(1980)**

Suppose $E$ is a co-$\kappa$-Suslin equivalence on $\mathbb{R}$ and $E$ is in a universe of ZF plus the axiom of determinacy. Then in that universe, either

- $\mathbb{R}/E$ embeds into $\kappa$ or
- $\mathbb{R}$ embeds into $\mathbb{R}/E$.

These choices are mutually exclusive.

This proof uses infintary logic as its main tool. A collection of statements controlled by $L[T]$ replaces $\Sigma_1^1$.

- $E$ is still an equivalence relation after Cohen-forcing if $E$ is inside of a universe ZF plus the axiom of determinacy.
- Universes that satisfy the axiom of choice and have large cardinals contain universes of determinacy
- Suslin sets naturally occur in those universes of determinacy.

# The Full Trichotomy from Determinacy

### A.Caicedo-R.Ketchersid(2011)

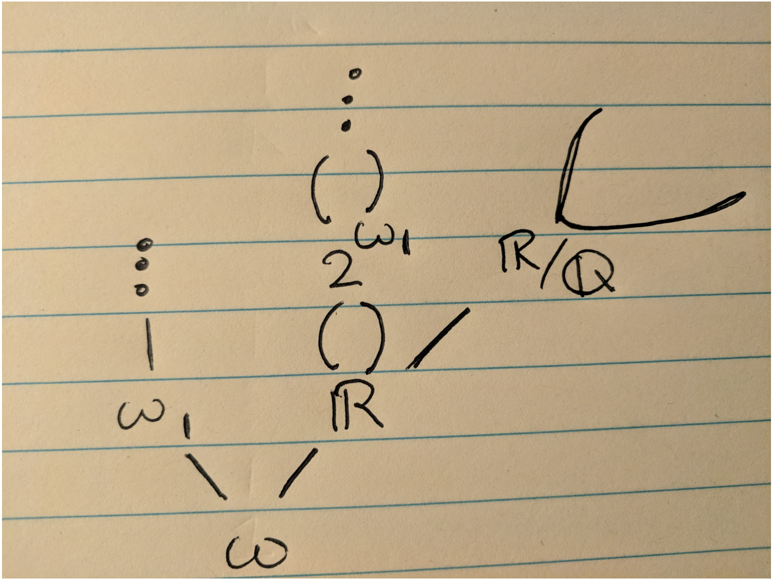
Suppose $E$ is an equivalence relation on $\mathbb{R}$ and $E$ is in a universe of ZF plus the axiom of determinacy. Then in that universe, either

- $\mathbb{R}/E$ is well-orderable, or
- $\mathbb{R}$ embeds into $\mathbb{R}/E$ and $\mathbb{R}/E$ embeds into $2^\alpha$ for some ordinal $\alpha$, or
- $\mathbb{R}/\mathbb{Q}$ embeds into $\mathbb{R}/E$.

These choices are mutually exclusive.

This proof replaces uses a combination of techniques from inner model theory and determinacy.

# The Full Trichotomy from Determinacy



# The Problem

- The Harrington-Shelah theorem doesn't have the full trichotomy.
- The Caicedo-Ketchersid result isn't as specific as it could be.
- The proof methods are extremely varied.

**Goal**: Prove the full trichotomy while retaining specificity and coherence of proof technique.

# Some Definitions

**Definition**

The infinity Borel codes are built up recursively, and so are their interpretations. Let $\{U_n : n \in \omega\}$ be a basis for the topology on $\mathbb{R}$.

- If $c = \langle 0, n \rangle$, then $A_c := U_n$,
- if $c = \langle 1, \langle c_\beta : \beta < \alpha \rangle \rangle$, then $A_c := \bigcap_{\beta<\alpha} A_{c_\beta}$,
- if $c = \langle 2, d \rangle$, then $A_c$ is the complement of $A_d$.

The collection of infinity Borel codes is denoted $\mathcal{B}_\infty$.

- Code length countable $\implies$ Borel
- "$A_c = \emptyset$" and "$A_c \subseteq A_d$" are only absolute for countable length codes.

# The Pointclass

Suppose $T$ is a tree whose nodes are finite strings from $\omega^n \times \kappa$. Let $\alpha$ be the least admissible ordinal for the structure $L_\alpha[T]$.

## Definition

A code $c$ is in $\Gamma$ iff $c = \langle 1, \langle c_\beta : \beta < \alpha \rangle \rangle$ where

- $\{c_\beta : \beta < \alpha\} \subseteq \mathcal{B}_\infty \cap L_\alpha[T]$, and
- $\{c_\beta : \beta < \alpha\}$ is $\Sigma_1(L_\alpha[T])$ definable.

$\Gamma$ corresponds to $\Sigma^1_1$ and $\mathcal{B}_\infty \cap L_\alpha[T]$ corresponds to $\Delta^1_1$.

## Proposition

$\Gamma$ is "closed" under finite unions, $\Sigma_1(L_\alpha[T])$ intersections, Cartesian products, and projections. $\Gamma$ also contains a code for $p[T]$.

# Pointclass Properties

## Definition

- For $c \in \Gamma$, say $c$ is **consistent** iff $A_c \neq \emptyset$ after collapsing $\kappa$ to $\omega$.
- For $c, d \in \Gamma$, say $c \leq d$ iff $A_c \subseteq A_d$ after collapsing $\kappa$ to $\omega$.

For $c, d \in \mathcal{B}_\infty \cap L_\alpha[T]$, these are $\Sigma_1(L_\alpha[T])$ statements.

## Barwise Compactness

Let $c \in \Gamma$. If $c$ is inconsistent, then there is a $\beta < \alpha$ so that $\langle 1, \langle c_\gamma : \gamma < \beta \rangle \rangle$ is inconsistent. In particular, if $c$ is a descending intersection, then some $c_\beta$ is inconsistent.

## Barwise Completeness

Let $c \in \Gamma$. Then $\{d \in \mathcal{B}_\infty \cap L_\alpha[T] : c \leq d\}$ is a $\Sigma_1(L_\alpha[T])$ subset of $L_\alpha[T]$.

# A Semantic Proof of the Harrington-Shelah Theorem

- Start with $E$ a co-$\kappa$-Suslin equivalence relation.
- See if the code for the intersection of the $\mathcal{B}_\infty \cap L_\alpha[T]$ sets which span multiple equivalence classes is consistent.
- If it is inconsistent, then $\mathcal{B}_\infty \cap L_\alpha[T]$ maps onto $\mathbb{R}/E$ and so $\mathbb{R}/E$ is countable.
- If is consistent, note that this code is in $\Gamma$. Force with the codes in $\Gamma$ below it.
- The generics for this forcing correspond to reals. Distinct generic reals are $E$-inequivalent.
- Build a perfect tree of generics. This embeds $\mathbb{R}$ into $\mathbb{R}/E$.

# The Extended Trichotomy

**A.Caicedo-R.Ketchersid(2011)**

Suppose $E$ is an equivalence relation on $\mathbb{R}$ and $E$ is in a universe of ZF plus the axiom of determinacy. Then in that universe, either

- $\mathbb{R}/E$ is well-orderable, or
- $\mathbb{R}$ embeds into $\mathbb{R}/E$ and $\mathbb{R}/E$ embeds into $2^\alpha$ for some ordinal $\alpha$, or
- $\mathbb{R}/\mathbb{Q}$ embeds into $\mathbb{R}/E$.

These choices are mutually exclusive.

# The Extended Trichotomy

## J. Holshouser

Suppose $E$ is a $\kappa$-Suslin and co-$\kappa$-Suslin equivalence relation, and $E$ is in a universe of ZF plus the axiom of determinacy. Then in that universe, either

- ${}^{\mathbb{R}}/_{E}$ embeds into $\kappa$, or
- $\mathbb{R}$ embeds into ${}^{\mathbb{R}}/_{E}$ and ${}^{\mathbb{R}}/_{E}$ embeds into $2^{\kappa}$, or
- ${}^{\mathbb{R}}/_{\mathbb{Q}}$ embeds into ${}^{\mathbb{R}}/_{E}$.

These choices are mutually exclusive.

# Proof Sketch

- Start with $E$ a $\kappa$ and co-$\kappa$-Suslin equivalence relation, witnessed by $T$ and $T'$.
- Generate an approximating equivalence relation: $x\tilde{E}y$ iff $x$ and $y$ are in the same $E$-invariant $\mathcal{B}_\infty \cap L_\alpha[T, T']$ sets.
- If $\tilde{E} = E$, then there is a map $\varphi : \mathbb{R} \to 2^\kappa$ so that $\varphi(x) = \varphi(y)$ iff $xEy$.
- Otherwise the difference between $\tilde{E}$ and $E$ is in $\Gamma$. Force below this.
- The generics for this forcing correspond to reals. Distinct generic reals are $E$-inequivalent.
- Build a link-structure of generics. This embeds $\mathbb{R}/\mathbb{Q}$ into $\mathbb{R}/E$.

# Questions

- Is $\Gamma = \Sigma^1_1$ for $T$ a computable tree?
- Can $\Gamma$ be used to figure out what kind of subset of $2^\kappa$ that $\mathbb{R}/E$ looks like?
- Can $\Gamma$ be used to extend other known dichotomies for Borel equivalence relations?
- If $\Gamma$ is used as a basis for a topology, what properties does that topology have?

# Thanks for Listening