Successes and Failures in Translating Strategies Across Selection Games

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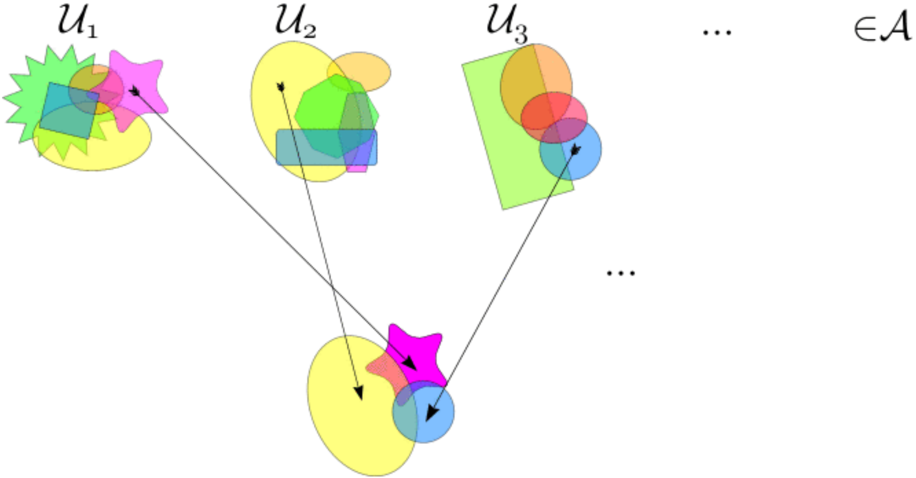
Three Topological Properties

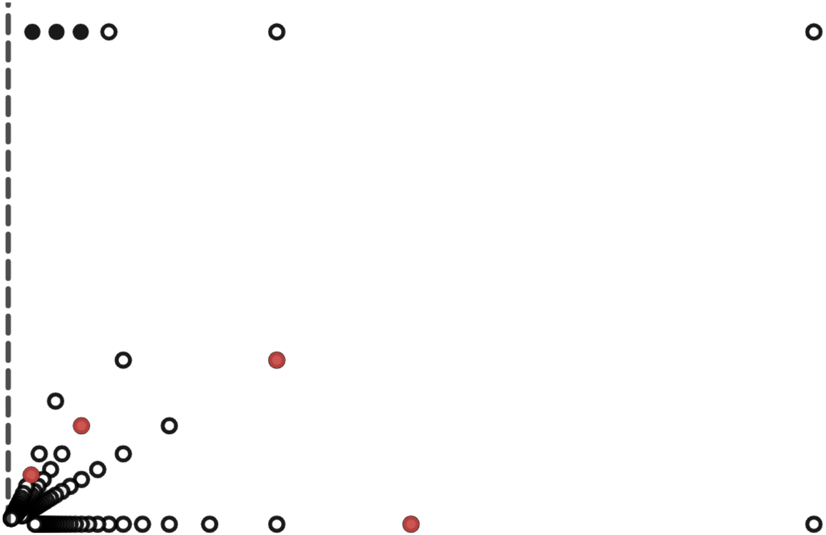
Suppose X is a topological space.

- **Lindelöf**: every open cover of X has a countable subcover.
- **Separable**: X has a countable dense subset.
- **Countable Tightness**: if x is in the closure of A, then there is a countable $B \subseteq A$ so that x is in the closure of B.

Rothberger



Strong Countable Fan Tightness



Selection Principles (Menger, Hurewicz 1924) (Scheepers 1996)

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are collections.

$S_1(\mathcal{A}, \mathcal{B})$

$S_1(\mathcal{A}, \mathcal{B})$ means that for all sequences $\langle A_n \rangle$ consisting of elements of $\mathcal{A}$, there are choices $x_n \in A_n$ so that $\{x_n : n \in \omega\} \in \mathcal{B}$.

$S_{\text{fin}}(\mathcal{A}, \mathcal{B})$

$S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ means that for all sequences $\langle A_n \rangle$ consisting of elements of $\mathcal{A}$, there are finite $F_n \subseteq A_n$ so that $\bigcup_n F_n \in \mathcal{B}$.

Notation

- Let $\mathcal{O}(X)$ denote the open covers of X.
- Let $\mathscr{D}_X$ denote the dense subsets of X.
- Let $\Omega_{X,x}$ denote the sets $A \subseteq X$ such that $x \in \overline{A}$.

Note that

- X is Rothberger if and only if $S_1(\mathcal{O}(X), \mathcal{O}(X))$.
- X is selectively separable if and only if $S_{fin}(\mathscr{D}_X, \mathscr{D}_X)$.
- X has strong countable fan tightness at x if and only if $S_1(\Omega_{X,x}, \Omega_{X,x})$.

Strategies

- A **perfect information (PI) strategy** for either player One or Two is a strategy for responding to the other player that takes as inputs all of the previous plays of the game.
- A **Markov strategy** for player Two is a strategy that takes as inputs the current turn number and the most recent play of player One.
- A **pre-determined (PD) strategy** for player One is a strategy where the only input is the current turn number.
- A strategy is **winning** if following the strategy guarantees that the player will win the game.

Strategies

- Playing according to a PI strategy for One:

I	$\sigma(\emptyset)$	$\sigma(x_0)$	$\sigma(x_0, x_1)$	$\cdots$
II	x_0	x_1	x_2	$\cdots$

- Playing according to a PI strategy for Two:

I	A_0	A_1	A_2	$\cdots$
II	$\tau(A_0)$	$\tau(A_0, A_1)$	$\tau(A_0, A_1, A_2)$	$\cdots$

- Playing according to a Markov strategy for Two:

I	A_0	A_1	A_2	$\cdots$
II	$\tau(A_0, 0)$	$\tau(A_1, 1)$	$\tau(A_2, 2)$	$\cdots$

- Playing according to a PD strategy for One:

I	$\sigma(0)$	$\sigma(1)$	$\sigma(2)$	$\cdots$
II	x_0	x_1	x_2	$\cdots$

A Strength Hierarchy

For the selection game $G_1(\mathcal{A}, \mathcal{B})$:

$$\begin{array}{rcl} \text{Two has a winning Markov Strategy} & & \\ \Downarrow & & \\ \text{Two has a winning PI strategy} & & \\ \Downarrow & & \\ \text{One has no winning PI strategy} & & \\ \Downarrow & & \\ \text{One has no winning PD strategy} & \iff & S_1(\mathcal{A}, \mathcal{B}) \end{array}$$

Game Equivalence

Definition

Define $G_1(\mathcal{A}, \mathcal{C}) \leq_{\mathrm{II}} G_1(\mathcal{B}, \mathcal{D})$ as the conjunction of the following implications.

1. Two has Mark in $G_1(\mathcal{A}, \mathcal{C}) \implies$ Two has a Mark in $G_1(\mathcal{B}, \mathcal{D})$
2. Two has PI strat in $G_1(\mathcal{A}, \mathcal{C}) \implies$ Two has a PI strat in $G_1(\mathcal{B}, \mathcal{D})$
3. One has no PI strat in $G_1(\mathcal{A}, \mathcal{C}) \implies$ One has no PI strat in $G_1(\mathcal{B}, \mathcal{D})$
4. One has no PD strat in $G_1(\mathcal{A}, \mathcal{C}) \implies$ One has no PD strat in $G_1(\mathcal{B}, \mathcal{D})$

- This relation is transitive.
- if $G_1(\mathcal{A}, \mathcal{C}) \leq_{\mathrm{II}} G_1(\mathcal{B}, \mathcal{D})$ and $G_1(\mathcal{B}, \mathcal{D}) \leq_{\mathrm{II}} G_1(\mathcal{A}, \mathcal{C})$, we say the two games are **equivalent** and write $G_1(\mathcal{A}, \mathcal{C}) \equiv G_1(\mathcal{B}, \mathcal{D})$.
- There is a G_{fin} version of all of this.

Scheeper's Monotoncity Law

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be collections.

Monotonicity

- If $\mathcal{B} \subseteq \mathcal{A}$, then $G_\square(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_\square(\mathcal{B}, \mathcal{C})$.
- If $\mathcal{D} \subseteq \mathcal{C}$, then $G_\square(\mathcal{A}, \mathcal{D}) \leq_{\text{II}} G_\square(\mathcal{A}, \mathcal{C})$.

As an application of this law, $G_1(\Omega_{X,x}, \Omega_{X,x}) \leq_{\text{II}} G_1(\mathcal{D}_X, \Omega_{X,x})$ for any space X and point $x \in X$.

Related Spaces

We will focus on connections between games played on a space X and games played on a space Y that was built out of X. In particular, we look at when Y is

- the space of closed subsets of X, and
- the space of continuous functions $f : X \to \mathbb{R}$.

These constructions lines up with multiple natural topologies, and we can work with a fair amount of them.

The Fell Topology on the Closed Subsets of X

Let $\mathbb{F}(X)$ denote the collection of closed subsets of X.

The Upper Fell Topology (Fell 1962)

The **Upper Fell Topology** on $\mathbb{F}(X)$ is generated by basic open sets of the form

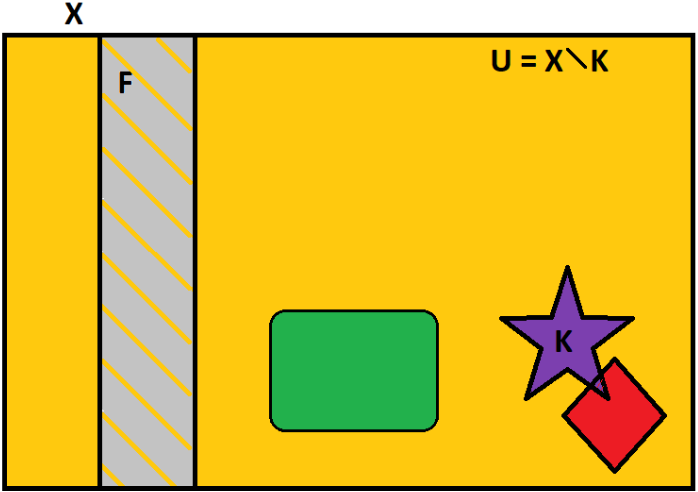
$$(U)^+ := \{F \in \mathbb{F}(X) : F \subseteq U\}$$

where U is the complement of a compact set.

A basic neighborhood of F has the form $(X \setminus K)^+$ where K is compact and $F \cap K = \emptyset$.
Let $\mathbb{F}^+(X)$ be the closed subsets of X with the upper Fell topology.

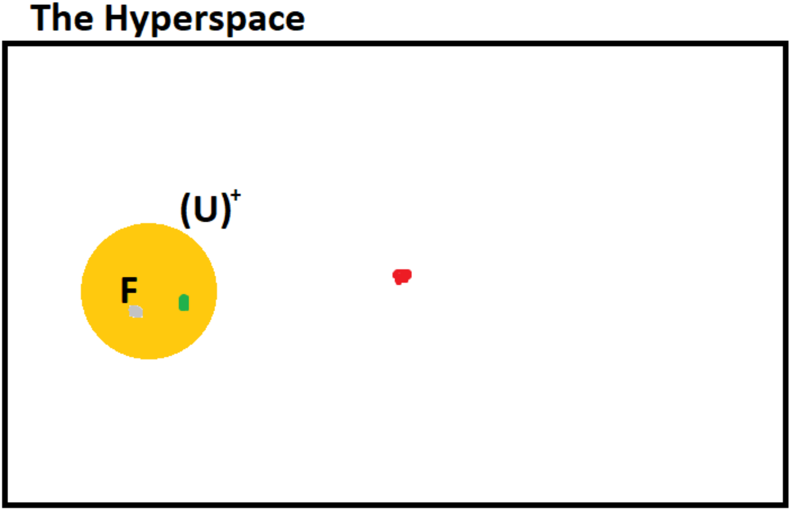
The Fell Topology on the Closed Subsets of X

F with an open neighborhood $(X \setminus K)^+$.



The Fell Topology on the Closed Subsets of X

F with an open neighborhood $(X \setminus K)^+$.



A Topological Connection

- $U \subseteq X$ is open if and only if $X \setminus U$ is a point in $\mathbb{F}^+(X)$.
- If $D \subseteq \mathbb{F}^+(X)$ is dense, then $\{X \setminus F : F \in D\}$ is an open cover of X.
- In fact, if $K \subseteq X$ is compact, then there is an $F \in D$ so that $K \subseteq X \setminus F$.

k-Covers

A non-trivial open cover $\mathscr{U}$ of X is a k**-cover** if for all compact $K \subseteq X$, there is a $U \in \mathscr{U}$ so that $K \subseteq U$.
$\mathcal{K}(X)$ is the collection of k-covers of X.

Proposition

- $\mathscr{U} \in \mathcal{K}(X)$ if and only if $\{X \setminus U : U \in \mathscr{U}\} \in \mathscr{D}_{\mathbb{F}^+(X)}$.
- $D \in \mathscr{D}_{\mathbb{F}^+(X)}$ if and only if $\{X \setminus F : F \in D\} \in \mathcal{K}(X)$.

Bijective Translation

Caruvana, Holshouser 2020

Assume that $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are collections and that $\bigcup \mathcal{C} \subseteq \bigcup \mathcal{A}$ and $\bigcup \mathcal{D} \subseteq \bigcup \mathcal{B}$. Suppose that there is a bijection $\beta : \bigcup \mathcal{A} \to \bigcup \mathcal{B}$ so that

- $A \in \mathcal{A}$ if and only if $\beta[A] \in \mathcal{B}$, and
- $C \in \mathcal{C}$ if and only if $\beta[C] \in \mathcal{D}$.

Then $G_\square(\mathcal{A}, \mathcal{C}) \equiv G_\square(\mathcal{B}, \mathcal{D})$.

Using Bijective Translation

Caruvana, Holshouser 2020

$G_\square(\mathcal{K}(X), \mathcal{K}(X)) \equiv G_\square(\mathcal{D}_{\mathbb{F}^+(X)}, D_{\mathbb{F}^+(X)})$.

We can prove this by defining $\beta : \mathcal{T}_X \to \mathbb{F}^+(X)$ as $\beta(U) = X \setminus U$ and applying the proposition.

Using Bijective Translation

Caruvana, Holshouser 2020

$G_\square(\mathcal{K}(X), \mathcal{K}(X)) \equiv G_\square(\mathcal{D}_{\mathbb{F}^+(X)}, D_{\mathbb{F}^+(X)})$.

We can prove this by defining $\beta : \mathcal{T}_X \to \mathbb{F}^+(X)$ as $\beta(U) = X \setminus U$ and applying the proposition.

Following Li's definition of $\mathcal{K}_F$-covers and letting $\mathbb{F}(X)$ denote the full Fell topology, we can also prove that $G_\square(\mathcal{K}_F(X), \mathcal{K}_F(X)) \equiv G_\square(\mathcal{D}_{\mathbb{F}(X)}, D_{\mathbb{F}(X)})$ using the same map β.

The Space of Continuous Functions

Set $C(X)$ to be collection of all continuous functions $f : X \to \mathbb{R}$.

The Topology of Point-Wise Convergence

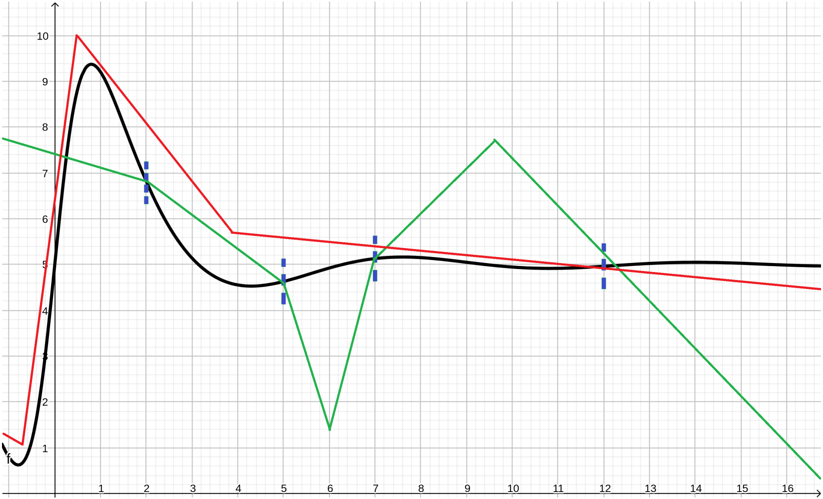
$C(X)$ with this topology will be denoted $C_p(X)$. The open sets are generated by sets of the form:

$$[f; \{x_0, \cdots, x_n\}, \varepsilon] = \{g : |f(x_0) - g(x_0)| < \varepsilon, \cdots, |f(x_n) - g(x_n)| < \varepsilon\}$$

where f is continuous, $x_0, \cdots, x_n \in X$, and $\varepsilon > 0$.

Common Topologies on the Space of Continuous Functions

f with a neighborhood $[f; F, \varepsilon]$.



A Topological Connection

- If $f : X \to \mathbb{R}$ is continuous, and $n \in \omega$, then $f^{-1}[(-2^{-n}, 2^{-n})]$ is open.
- If $A \subseteq C_p(X)$ has $\mathbf{0}$ in its closure, then for a fixed n,

$$\mathscr{U} = \{f^{-1}[(-2^{-n}, 2^{-n})] : f \in A\}$$

  is an open cover of X.
- In fact, if $F \subseteq X$ is finite, then there is a $U \in \mathscr{U}$ so that $F \subseteq U$.

ω-Covers

A non-trivial open cover $\mathscr{U}$ of X is an ω**-cover** if for all finite $F \subseteq X$, there is a $U \in \mathscr{U}$ so that $F \subseteq U$.
$\Omega(X)$ is the collection of ω-covers of X.

Proposition

If $A \in \Omega_{C_p(X),\mathbf{0}}$, then

$$\mathscr{U}(A,n) := \{f^{-1}[(-2^{-n},2^{-n})] : f \in A\} \in \Omega(X)$$

Proof.

Suppose $A \in \Omega_{C_p(X),\mathbf{0}}$. Let $F \subseteq X$ be finite. Then $[\mathbf{0}; F, 2^{-n}]$ is an open nhood of $\mathbf{0}$. So there is an $f \in A$ so that $f \in [\mathbf{0}; F, 2^{-n}]$. Then $F \subseteq f^{-1}[(-2^{-n},2^{-n})]$.

□

ω-Covers

Proposition

Suppose $f_n \in C_p(X)$ are so that

$$\{f_n^{-1}[(-2^{-n}, 2^{-n})] : n \in \omega\} \in \Omega(X).$$

Then $\{f_n : n \in \omega\} \in \Omega_{C_p(X),\mathbf{0}}$.

Proof.

Consider a basic open nhood $[\mathbf{0}; F, \varepsilon]$. Since F is finite, there is an n so that $2^{-n} < \varepsilon$ and $F \subseteq f_n^{-1}[(-2^{-n}, 2^{-n})]$. Thus $f_n \in [\mathbf{0}; F, \varepsilon]$. □

Turn Based Translation

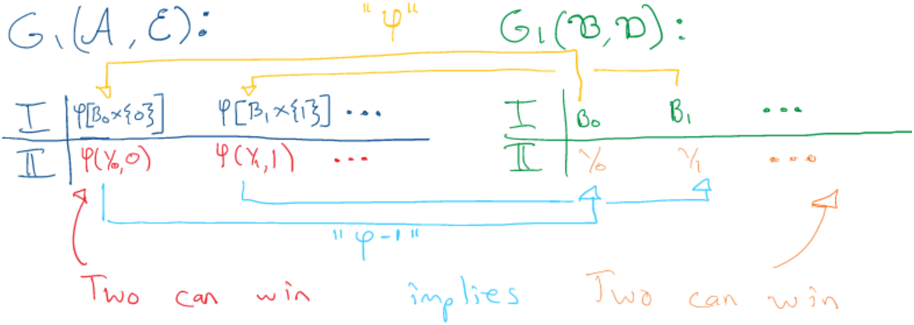
Caruvana, Holshouser 2020

Assume that $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are collections and that $\bigcup \mathcal{C} \subseteq \bigcup \mathcal{A}$ and $\bigcup \mathcal{D} \subseteq \bigcup \mathcal{B}$. Suppose that there is a function $\varphi : \bigcup \mathcal{B} \times \omega \to \bigcup \mathcal{A}$ so that

- If $B \in \mathcal{B}$ and $n \in \omega$, then $\varphi[B \times \{n\}] \in \mathcal{A}$, and
- If $y_n \in B_n$ and $\{\varphi(y_n, n) : n \in \omega\} \in \mathcal{C}$, then $\{y_n : n \in \omega\} \in \mathcal{D}$.

Then $G_1(\mathcal{A}, \mathcal{C}) \leq_{\mathrm{II}} G_1(\mathcal{B}, \mathcal{D})$.

Turn Based Translation



① $B \in \mathcal{B}$ and $n \in \omega \Rightarrow \varphi[B \times \{n\}] \in \mathcal{A}$

② $y_n \in B_n$ and $\{\varphi(y_n, n) : n \in \omega\} \in \mathcal{C} \Rightarrow \{y_n : n \in \omega\} \in \mathcal{D}$

Using Turn Based Translation

Caruvana, Holshouser 2020

$G_1(\Omega(X), \Omega(X)) \leq_{\mathrm{II}} G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$.

We can prove this by defining $\varphi : C_p(X) \times \omega \to \mathcal{T}_X$ as $\varphi(f, n) = f^{-1}[(-2^{-n}, 2^{-n})]$ and applying our facts about ω-covers.

Using Turn Based Translation

Caruvana, Holshouser 2020

$G_1(\Omega(X), \Omega(X)) \leq_{\text{II}} G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}})$.

We can prove this by defining $\varphi : C_p(X) \times \omega \to \mathcal{T}_X$ as $\varphi(f, n) = f^{-1}[(-2^{-n}, 2^{-n})]$ and applying our facts about ω-covers.

This same φ shows $G_1(\mathcal{K}(X), \mathcal{K}(X)) \leq_{\text{II}} G_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$.

Complete Regularity

Definition

Recall that space X is **completely regular** if for every point $x \in X$ and closed set $F \subseteq X$ with $x \notin F$, there is a continuous function $f : X \to [0,1]$ so that $f(x) = 0$ and $f|_F = \mathbf{1}$.

Note that you can also find continuous functions to separate finite sets from closed sets, and even separate compact sets from closed sets.

Complete Regularity

Suppose that X is Hausdorff and completely regular.

Proposition

If $\mathscr{U} \in \Omega(X)$, then

$$A(\mathscr{U}) := \{f : (\exists U \in \mathscr{U})[f|_{X \setminus U} = \mathbf{1}]\} \in \Omega_{C_p(X),\mathbf{0}}.$$

Proof.

Suppose $\mathscr{U} \in \Omega(X)$. Consider a basic open nhood $[\mathbf{0}; F, \varepsilon]$. There is a $U \in \mathscr{U}$ so that $F \subseteq U$. There is then a continuous $f : X \to \mathbb{R}$ so that $f|_F = \mathbf{0}$ and $f|_{X \setminus U} = \mathbf{1}$. Then $f \in [\mathbf{0}; F, \varepsilon]$. □

Complete Regularity

Proposition

Suppose $\{f_n : n \in \omega\} \in \Omega_{C_p(X),\mathbf{0}}$ and that $U_n \subseteq X$ are open so that $f_n|_{X \setminus U_n} = \mathbf{1}$. Then $\{U_n : n \in \omega\} \in \Omega(X)$.

Proof.

Suppose $F \subseteq X$ is finite. Then $[\mathbf{0}; F, 1]$ is open nhood of $\mathbf{0}$, so there is an n so that $f_n \in [\mathbf{0}; F, 1]$. Thus $F \subseteq f_n^{-1}[(-1, 1)]$. $f_n|_{X \setminus U_n} = \mathbf{1}$, this means that $F \cap (X \setminus U_n) = \emptyset$. Therefore $F \subseteq U_n$. □

General Translation

Theorem (Caruvana and Holshouser 2019)

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be collections. Suppose there are functions

- $\overleftarrow{T}_{\mathrm{I},n} : \mathcal{B} \to \mathcal{A}$ and
- $\overrightarrow{T}_{\mathrm{II},n} : \bigcup \mathcal{A} \times \mathcal{B} \to \bigcup \mathcal{B}$

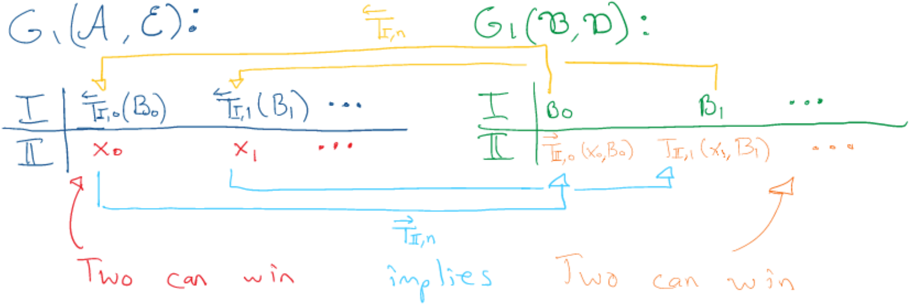
so that

1. if $x \in \overleftarrow{T}_{\mathrm{I},n}(B)$, then $\overrightarrow{T}_{\mathrm{II},n}(x, B) \in B$ and
2. if $x_n \in \overleftarrow{T}_{\mathrm{I},n}(B_n)$ for all n, then

$$\{x_n : n \in \omega\} \in \mathcal{C} \implies \left\{\overrightarrow{T}_{\mathrm{II},n}(x_n, B_n) : n \in \omega\right\} \in \mathcal{D}$$

Then $G_1(\mathcal{A}, \mathcal{C}) \leq_{\mathrm{II}} G_1(\mathcal{B}, \mathcal{D})$.

General Translation



1. $x \in \overleftarrow{T}_{\mathrm{I},n}(B) \Rightarrow \overrightarrow{T}_{\mathrm{II},n}(x, B) \in B$
2. $x_n \in \overleftarrow{T}_{\mathrm{I},n}(B_n)$ and $\{x_n : n \in \omega\} \in \mathcal{E} \Rightarrow \{\overrightarrow{T}_{\mathrm{II},n}(x_n, B_n) : n \in \omega\} \in \mathcal{D}$

Using General Translation

Let X be a Hausdorff completely regular space.

Theorem (Caruvana and Holshouser 2019)

$G_1(\Omega_{C_p(X),\mathbf{0}}, \Omega_{C_p(X),\mathbf{0}}) \leq_{\mathrm{II}} G_1(\Omega(X), \Omega(X))$

Define $\overleftarrow{T}_{\mathrm{I},n}(\mathscr{U}) = A(\mathscr{U})$ and set $\overrightarrow{T}_{\mathrm{II},n}(f, \mathscr{U})$ to be a choice of $U \in \mathscr{U}$ so that $f[X \setminus U] = \{1\}$ if possible, and X otherwise. Confirm that

- $\overleftarrow{T}_{\mathrm{I},n}$ and $\overrightarrow{T}_{\mathrm{II},n}$ are well-defined,
- If $f \in \overleftarrow{T}_{\mathrm{I},n}(\mathscr{U})$, then $\overrightarrow{T}_{\mathrm{II},n}(f, \mathscr{U}) \in \mathscr{U}$, and
- If $\mathscr{U}_n$ are ω-covers, f_n and $U_n \in \mathscr{U}_n$ are so that $f_n[X \setminus U_n] = \{1\}$, and $\mathbf{0} \in \overline{\{f_n : n \in \omega\}}$, then $\{U_n : n \in \omega\}$ is an ω-cover.

A Game Reduction That Does Not Use Translation

Theorem

$G_1(\Omega, \Omega) \leq_{\mathrm{II}} G_1(\mathcal{O}, \mathcal{O})$.

- Player II has a winning (Markov) strategy in $G_1(\Omega, \Omega)$ if and only if they have a winning (Markov) strategy in $G_1(\mathcal{O}, \mathcal{O})$.
- Using a result of Pawlikowski, Player I has a winning strat in $G_1(\mathcal{O}, \mathcal{O})$ if and only if they have winning PD strat.
- Using the generalization of Pawlikowski, Player I has a winning strat in $G_1(\Omega, \Omega)$ if and only if they have winning PD strat.
- Using the bijection between ω and ω^2, $S_1(\Omega, \Omega)$ is equivalent $S_1(\mathcal{O}, \mathcal{O})$.

Brittle Game Reductions

- The game reductions that came from translations all generalize to different ideals. In particular, they work both in reference to compact sets and to finite sets.
- $G_1(\Omega, \Omega) \leq_{\text{II}} G_1(\mathcal{O}, \mathcal{O})$, but $G_1(\mathcal{K}, \mathcal{K}) \not\leq_{\text{II}} G_1(\mathcal{O}, \mathcal{O})$.

Thanks for Listening