Limited Information Strategies and Discrete Selectivity

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Pure Math Seminar

Function Space

Suppose X is a topological space.

- $C_p(X)$ is the space of continuous functions $f : X \to \mathbb{R}$
- It is endowed with the topology of pointwise convergence.

Function Space

Suppose X is a topological space.

- $C_p(X)$ is the space of continuous functions $f : X \to \mathbb{R}$
- It is endowed with the topology of pointwise convergence.
- It will be convenient to assume that X is Hausdorff and completely regular (for every closed $C \subseteq X$ and $x \in X \setminus C$, there is a continuous function $f : X \to \mathbb{R}$ so that $f(x) = 0$ and $f[C] = \{1\}$).

Open Neighborhoods of $C_p(X)$

To define the topology on $C_p(X)$ using open sets, we use the following neighborhoods:

Definition

If $f : X \to \mathbb{R}$ is continuous, $F = \{x_1, \cdots, x_n\} \subseteq X$, and $\varepsilon > 0$, then

$$[f, F, \varepsilon] = \{g \in C_p(X) : |g(x_i) - f(x_i)| < \varepsilon \text{ for } 1 \leq i \leq n\}$$

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$$[f, F, \varepsilon] = \{g \in C_p(X) : |g(x_i) - f(x_i)| < \varepsilon \text{ for } 1 \leq i \leq n\}$$

Varying over F and ε forms a neighborhood basis at f.

Closed Discrete Selections

The concept of a closed discrete selection was isolated by Sanchez and Tkachuk in 2017.

Closed Discrete Selection

For every sequence $(U_n : n \in \omega)$ of open subsets of X, there are points $x_n \in U_n$ so that $\{x_n : n \in \omega\}$ is closed and discrete.

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Note that if X is first countable (or even has one point with a countable neighborhood basis), then it cannot have closed discrete selection. The converse is not generally true.

Closed Discrete Selections of Functions

Tkachuk, 2017

Suppose X is Hausdorff and completely regular. Then the following are equivalent:

- $C_p(X)$ fails to satisfy closed discrete selection
- X is countable
- $C_p(X)$ is first-countable

Proof

If $C_p(X)$ fails to satisfy closed discrete selection, then X is countable:

- Suppose X is uncountable.
- Consider a sequence $[f_n, F_n, \varepsilon]$ of basic open sets in $C_p(X)$.
- Since X is uncountable, we can find a point $x^* \in X \setminus F_n$.
- We can find continuous functions g_n so that $g_n|_{F_n} = f_n$ and $g_n(x^*) = n$.
- Then $g_n \in [f_n, F_n, \varepsilon]$ and $\{g_n : n \in \omega\}$ is closed discrete.

Selection Principles

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are collections of sets.

$S_1(\mathcal{A}, \mathcal{B})$

For every sequence $(A_n : n \in \omega)$ of sets from $\mathcal{A}$, there are $C_n \in A_n$ so that $\{C_n : n \in \omega\} \in \mathcal{B}$

$S_{FIN}(\mathcal{A}, \mathcal{B})$

For every sequence $(A_n : n \in \omega)$ of sets from $\mathcal{A}$, there are finite $F_n \subseteq A_n$ so that $\bigcup_n F_n \in \mathcal{B}$

Common Selection Principles

Let $\mathcal{O}$ be the open covers of X. Then $S_1(\mathcal{O}, \mathcal{O})$ and $S_{FIN}(\mathcal{O}, \mathcal{O})$ are both strengthenings of the Lindelof property.

- $S_1(\mathcal{O}, \mathcal{O})$ is called the Rothberger property.
- $S_{FIN}(\mathcal{O}, \mathcal{O})$ is called the Menger property.

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- $S_1(\mathcal{O},\mathcal{O})$ is called the Rothberger property.
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$$S_1(\mathcal{O},\mathcal{O}) \Rightarrow S_{FIN}(\mathcal{O},\mathcal{O}) \Rightarrow \text{Lindelof}$$

and

$$\text{Compact} \Rightarrow \sigma - Compact \Rightarrow S_{FIN}(\mathcal{O},\mathcal{O}) \Rightarrow \text{Lindelof}$$

$S_\square(\mathcal{A}, \mathcal{B})$ can be turned into a two-player game.

- The game is played over rounds indexed by the naturals.
- At round n, player I plays a set A_n from $\mathcal{A}$ and player II responds by playing a selection C_n from A_n.
- If those selections are singletons, then the game is $G_1(\mathcal{A}, \mathcal{B})$. If they are finite sets, then the game is $G_{FIN}(\mathcal{A}, \mathcal{B})$.
- Player II wins a given run of the game $(A_0, C_0, A_1, C_1, \cdots)$ if $\bigcup_n C_n \in \mathcal{B}$.
- If player II does not win, then player I wins.

Perfect Information Strategies

Fix a game $G_{\square}(\mathcal{A}, \mathcal{B})$.

- A perfect information strategy for player I takes in a run of $G_{\square}(\mathcal{A}, \mathcal{B})$ up to some round n and outputs a set $A_{n+1} \in \mathcal{A}$.
- A perfect information strategy for player II takes in a run of $G_{\square}(\mathcal{A}, \mathcal{B})$ up to some round n and outputs a selection C_n from I's most recent move.

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- A perfect information strategy for player II takes in a run of $G_{\square}(\mathcal{A}, \mathcal{B})$ up to some round n and outputs a selection C_n from I's most recent move.
- A strategy σ for player I is winning if the run $(\sigma(\emptyset), C_0, \sigma(C_0), C_1, \cdots)$ wins for I no matter what selections II makes. If I has a winning strategy we write $I \uparrow G_{\square}(\mathcal{A}, \mathcal{B})$.
- A strategy τ for player II is winning if the run $(A_0, \tau(A_0), A_1, \tau(A_0, A_1), \cdots)$ wins for II no matter what sets player I plays. If II has a winning strategy we write $II \uparrow G_{\square}(\mathcal{A}, \mathcal{B})$.

Limited Information Strategies

- A Markov tactic for II is a strategy $\tau(A, n)$ which takes in only the round number and the most recent move of I. If II has a winning Markov tactic we write $II \uparrow_{mark} G_{\square}(\mathcal{A}, \mathcal{B})$.
- A pre-determined strategy for I is a strategy $\sigma(n)$ which takes in only the round number. If I has a winning pre-determined strategy we write $I \uparrow_{pre} G_{\square}(\mathcal{A}, \mathcal{B})$.

Easy Implications

$$I \uparrow_{pre} G(\mathcal{A},\mathcal{B}) \Rightarrow I \uparrow G(\mathcal{A},\mathcal{B}) \Rightarrow II \not\uparrow G(\mathcal{A},\mathcal{B})$$

$$II \uparrow_{mark} G(\mathcal{A},\mathcal{B}) \Rightarrow II \uparrow G(\mathcal{A},\mathcal{B}) \Rightarrow I \not\uparrow G(\mathcal{A},\mathcal{B})$$

$$II \uparrow G(\mathcal{A},\mathcal{B}) \Rightarrow S(\mathcal{A},\mathcal{B})$$

$$I \not\uparrow_{pre} G(\mathcal{A},\mathcal{B}) \iff S(\mathcal{A},\mathcal{B})$$

Useful Selection Games

- $G_1(\mathcal{O}, \mathcal{O})$ is the Rothberger game
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Useful Selection Games

- $G_1(\mathcal{O}, \mathcal{O})$ is the Rothberger game
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- The point-open game is traditionally played with player I choosing points x_n and player II choosing open sets $U_n \ni x_n$. Player I wins if $\{U_n : n \in \omega\}$ is an open cover.
- Cast as a selection game, this looks like $G_1(\mathcal{P}, \neg\mathcal{O})$, where $\mathcal{P}$ is the collection of point bases and $\neg\mathcal{O}$ consists of sequences of opens sets which are not open covers.
- The closed discrete selection game is $G_1(\mathcal{T}, CD)$, where $\mathcal{T}$ is the collection of open sets and CD is the collection of closed discrete subsets of X.

Useful Selection Games

- $I \uparrow G_1(\mathcal{P}, \neg\mathcal{O})$ if and only if $II \uparrow G_1(\mathcal{O}, \mathcal{O})$
- $II \uparrow G_1(\mathcal{P}, \neg\mathcal{O})$ if and only if $I \uparrow G_1(\mathcal{O}, \mathcal{O})$
- $I \uparrow_{pre} G_1(\mathcal{P}, \neg\mathcal{O})$ if and only if X is countable.

Proof

$I \uparrow G_1(\mathcal{P}, \neg\mathcal{O})$ implies $II \uparrow G_1(\mathcal{O}, \mathcal{O})$:

- Let σ be a winning strategy for I in $G_1(\mathcal{P}, \neg\mathcal{O})$.
- Say $\sigma(\emptyset) = x_0$.
- We need to define a strategy τ for II in $G_1(\mathcal{O}, \mathcal{O})$.
- Suppose $\mathcal{U}_0$ has been played by I in $G_1(\mathcal{O}, \mathcal{O})$. Then we can find a $U_0 \in \mathcal{U}$ so that $x_0 \in \mathcal{U}_0$. Set $\tau(\mathcal{U}_0) = U_0$.
- Pretend II has repsonded in $G_1(\mathcal{P}, \neg\mathcal{O})$ with U_0. Set $x_1 = \sigma(x_0, U_0)$.
- Now suppose I plays $\mathcal{U}_1$ in $G_1(\mathcal{O}, \mathcal{O})$ in response to $\tau(\mathcal{U}_0)$. Then there is a $U_1 \in \mathcal{U}_1$ so that $x_1 \in U_1$. Set $\tau(\mathcal{U}_1) = U_1$.
- Continue in this way to inductively define τ. Because σ is winning for I in $G_1(\mathcal{P}, \neg\mathcal{O})$, $\{\tau(\mathcal{U}_n) : n \in \omega\}$ will be an open cover.

Closed Discrete Selections of Functions II

Tkachuk, 2017

Suppose X is Hausdorff and completely regular. Then the following are equivalent:

- $C_p(X)$ fails to satisfy closed discrete selection
- X is countable
- $C_p(X)$ is first-countable

Equivalently, in the language of games:

Tkachuk, 2017

Suppose X is Hausdorff and completely regular. Then the following are equivalent:

- $I \uparrow_{pre} G_1(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$
- $I \uparrow_{pre} G_1(\mathcal{P}_X, \neg\mathcal{O}_X)$
- $C_p(X)$ is first-countable

Closed Discrete Selections of Functions II

Tkachuk was also able to prove a partial result for perfect information strategies:

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Also, if $II \uparrow G(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$, then $II \uparrow G_1(\mathcal{P}_X, \neg\mathcal{O}_X)$.

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Also, if $II \uparrow G(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$, then $II \uparrow G_1(\mathcal{P}_X, \neg\mathcal{O}_X)$.

Is it true that if $II \uparrow G_1(\mathcal{P}_X, \neg\mathcal{O}_X)$, then $II \uparrow G(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$?

ω-Covers and Games

Consider a topological space $(X, \mathcal{T})$.

- $\mathcal{U}$ is an ω-cover of X if whenever $F \subseteq X$ is finite, there is a $U \in \mathcal{U}$ so that $F \subseteq U$.
- Let Ω_X be the collection of ω-covers of X.
- For $F \subseteq X$ finite, let $\mathcal{N}[F] = \{U \in \mathcal{T} : F \subseteq U\}$. We can then consider $\mathcal{F}_X = \{\mathcal{N}[F] : F \subseteq X \text{ is finite}\}$.

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- We can now define the finite-open game $G_1(\mathcal{F}, \neg\mathcal{O})$ and its variant $G_1(\mathcal{F}, \neg\Omega)$
- We can also define $G_1(\Omega, \Omega)$.

Point-Open Versus Finite Open

Proposition

The following are equivalent:

- $I \uparrow G_1(\mathcal{P}, \neg\mathcal{O})$
- $I \uparrow G_1(\mathcal{F}, \neg\mathcal{O})$
- $I \uparrow G_1(\mathcal{F}, \neg\Omega)$
- $II \uparrow G_1(\mathcal{O}, \mathcal{O})$
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- $I \uparrow G_1(\mathcal{F}, \neg\Omega)$
- $II \uparrow G_1(\mathcal{O}, \mathcal{O})$
- $II \uparrow G_1(\Omega, \Omega)$

What Tkachuk actually showed was that $I \uparrow G_1(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$ if and only if $I \uparrow G_1(\mathcal{F}, \neg\Omega)$ and that if $II \uparrow G_1(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$, then $II \uparrow G_1(\mathcal{F}, \neg\Omega)$.

Point-Open Versus Finite Open II

Proposition

- If $II \uparrow G_1(\mathcal{F}, \neg\Omega)$, then $II \uparrow G_1(\mathcal{P}, \neg\mathcal{O})$.
- $G_1(\mathcal{F}, \neg\Omega)$ and $G_1(\Omega, \Omega)$ are dual.
- It is consistent with ZFC that there is a space X where $I \uparrow G_1(\mathcal{O}, \mathcal{O})$ but $I \not\uparrow G_1(\Omega, \Omega)$
- Thus it is consistent with ZFC that there is a space X where $II \uparrow G_1(\mathcal{P}, \neg\mathcal{O})$, but $II \not\uparrow G_1(\mathcal{F}, \neg\Omega)$.

Point-Open Versus Finite Open II

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- Thus it is consistent with ZFC that there is a space X where $II \uparrow G_1(\mathcal{P}, \neg\mathcal{O})$, but $II \not\uparrow G_1(\mathcal{F}, \neg\Omega)$.

With this in mind, the proper question is: does $II \uparrow G_1(\mathcal{F}_X, \neg\Omega_X)$ imply $II \uparrow G(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$?

Answering Tkachuk's Question

Clontz/Holshouser 2018

The following are equivalent.

- $II \uparrow G_1(\mathcal{F}_X, \neg\Omega_X)$
- $I \uparrow G_1(\Omega_X, \Omega_X)$
- $II \uparrow G(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$
- $II \uparrow_{mark} G_1(\mathcal{F}_X, \neg\Omega_X)$
- $I \uparrow_{pre} G_1(\Omega_X, \Omega_X)$, i.e. X is not Rothberger with respect to Ω-covers
- $II \uparrow_{mark} G(\mathcal{T}_{C_p(X)}, CD_{C_p(X)})$

Function Space II

Suppose X is a topological space.

- $C_k(X)$ is the space of continuous functions $f : X \to \mathbb{R}$.
- It is endowed with the compact-open topology ("Uniform Convergence on Compact Sets").
- If $f : X \to \mathbb{R}$ is continuous, $K \subseteq X$ is compact, and $\varepsilon > 0$, then

$$[f, K, \varepsilon] = \{g \in C_p(X) : |g(x) - f(x)| < \varepsilon \text{ for all } x \in K\}$$

  These sets form a basis for the topology on $C_k(X)$.

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$$[f, K, \varepsilon] = \{g \in C_p(X) : |g(x) - f(x)| < \varepsilon \text{ for all } x \in K\}$$

  These sets form a basis for the topology on $C_k(X)$.

What happens when we play the closed discrete game on $C_k(X)$?

k-Covers and Games

Consider a topological space $(X, \mathcal{T})$.

- Let $K(X)$ be the space of compact subsets of X.
- $\mathcal{U}$ is a k-cover of X if whenever $K \subseteq X$ is compact, there is a $U \in \mathcal{U}$ so that $K \subseteq U$.
- Let $\mathcal{K}_X$ be the collection of k-covers of X.
- For $K \subseteq X$ compact, let $\mathcal{N}[K] = \{U \in \mathcal{T} : K \subseteq U\}$. We can now consider $\mathcal{N}[K(X)] = \{\mathcal{N}[K] : K \subseteq X \text{ is compact}\}$.

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- Let $\mathcal{K}_X$ be the collection of k-covers of X.
- For $K \subseteq X$ compact, let $\mathcal{N}[K] = \{U \in \mathcal{T} : K \subseteq U\}$. We can now consider $\mathcal{N}[K(X)] = \{\mathcal{N}[K] : K \subseteq X \text{ is compact}\}$.
- These sets are used to form the compact-open game $G_1(\mathcal{N}[K(X)], \neg\mathcal{O})$ and its variant $G_1(\mathcal{N}[K(X)], \neg\mathcal{K})$.
- We can also play the k-Rothberger game $G_1(\mathcal{K}, \mathcal{K})$.

Hemicompactness

Hemicompact

X is hemicompact if there are compact sets $K_n \subseteq X$ for $n \in \omega$ so that $X = \bigcup_n K_n$ and whenever $K \subseteq X$ is compact, there is some n so that $K \subseteq K_n$.

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Theorem

The following are equivalent:

- X is hemicompact
- $C_k(X)$ is first countable
- $I \uparrow_{pre} G_1(\mathcal{N}[K(X)], \neg\mathcal{K}_X)$

Clontz' Duality

Clontz 2018

If $\mathcal{R}$ is a reflection of $\mathcal{A}$, then the two games $G_1(\mathcal{A}, \mathcal{B})$ and $G_1(\mathcal{R}, \neg\mathcal{B})$ are dual.

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Caruvana/Holshouser 2018

Suppose $\mathcal{A}$ is a collection of subsets of X. For $A \in \mathcal{A}$, set $\mathcal{N}[A] = \{U \in \mathcal{T}_X : A \subseteq U\}$. Set

$$\mathcal{N}[\mathcal{A}] = \{\mathcal{N}[A] : A \in \mathcal{A}\}.$$

and

$$\mathcal{O}(X, \mathcal{A}) = \{\mathcal{U} \in \mathcal{O}_X : (\forall A \in \mathcal{A})(\exists U \in \mathcal{U})[A \subseteq U]\}.$$

Let $\mathcal{B}$ be a collection of open covers of X. Then $\mathcal{N}[\mathcal{A}]$ is a reflection of $\mathcal{O}(X, \mathcal{A})$ and therefore $G_1(\mathcal{N}[\mathcal{A}], \neg\mathcal{B})$ and $G_1(\mathcal{O}(X, \mathcal{A}), \mathcal{B})$ are dual.

Closed Discrete Selections of Functions III

Caruvana/Holshouser 2018

The following are equivalent.

- $I \uparrow G_1(\mathcal{N}[K(X)], \neg\mathcal{K}_X)$
- $II \uparrow G_1(\mathcal{K}_X, \mathcal{K}_X)$
- $I \uparrow G(\mathcal{T}_{C_k(X)}, CD_{C_k(X)})$

Closed Discrete Selections of Functions III

Caruvana/Holshouser 2018

The following are equivalent.

- $I \uparrow G_1(\mathcal{N}[K(X)], \neg\mathcal{K}_X)$
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Caruvana/Holshouser 2018

The following are equivalent.

- $I \uparrow_{pre} G_1(\mathcal{N}[K(X)], \neg\mathcal{K}_X)$
- $II \uparrow_{mark} G_1(\mathcal{K}_X, \mathcal{K}_X)$
- $I \uparrow_{pre} G(\mathcal{T}_{C_k(X)}, CD_{C_k(X)})$

Improving Strategies

Pawlikowski 1994

$I \uparrow_{pre} G_\square(\mathcal{O}, \mathcal{O})$ if and only if $I \uparrow G_\square(\mathcal{O}, \mathcal{O})$

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Caruvana/Holshouser 2018

Suppose $\mathcal{A} \subseteq \mathcal{B}$. Then $I \uparrow_{pre} G_{\square}(\mathcal{O}(X, \mathcal{A}), \mathcal{O}(X, \mathcal{B}))$ if and only if $I \uparrow G_{\square}(\mathcal{O}(X, \mathcal{A}), \mathcal{O}(X, \mathcal{B}))$

Closed Discrete Selections of Functions IV

Caruvana/Holshouser 2018

The following are equivalent.

- $II \uparrow G_1(\mathcal{N}[K(X)], \neg\mathcal{K}_X)$
- $I \uparrow G_1(\mathcal{K}_X, \mathcal{K}_X)$
- $II \uparrow G(\mathcal{T}_{C_k(X)}, CD_{C_k(X)})$
- $II \uparrow_{mark} G_1(\mathcal{N}[K(X)], \neg\mathcal{K}_X)$
- $I \uparrow_{pre} G_1(\mathcal{K}_X, \mathcal{K}_X)$, i.e. X is not Rothberger with respect to k-covers
- $II \uparrow_{mark} G(\mathcal{T}_{C_k(X)}, CD_{C_k(X)})$

Questions

- What happens with $C_p(X, [0,1])$?
- How far can Pawlikowski's result be generalized?
- How much of this theory can be recovered if the compact sets are replaced with a different ideal?
- How much of this theory can be recovered if instead of choosing subcovers we choose refinements?

Thanks for Listening