Successes and Failures in Translating Strategies Across Selection Games

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Selection Principles (Menger, Hurewicz 1924) (Scheepers 1996)

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are collections.

$S_1(\mathcal{A}, \mathcal{B})$

$S_1(\mathcal{A}, \mathcal{B})$ means that for all sequences A_n consisting of elements of $\mathcal{A}$, there are choices $x_n \in A_n$ so that $\{x_n : n \in \omega\} \in \mathcal{B}$.

$S_{\text{fin}}(\mathcal{A}, \mathcal{B})$

$S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ means that for all sequences A_n consisting of elements of $\mathcal{A}$, there are finite $F_n \subseteq A_n$ so that $\bigcup_n F_n \in \mathcal{B}$.

- Let $\mathcal{O}(X)$ denote the open covers of X.
- Let $\mathscr{D}_X$ denote the dense subsets of X.
- Let $\Omega_{X,x}$ denote the sets $A \subseteq X$ such that $x \in \overline{A}$.

Strategies

- A **perfect information (PI) strategy** for either player One or Two is a strategy for responding to the other player that takes as inputs all of the previous plays of the game.
- A **Markov strategy** for player Two is a strategy that takes as inputs the current turn number and the most recent play of player One.
- A **pre-determined (PD) strategy** for player One is a strategy where the only input is the current turn number.
- A strategy is **winning** if following the strategy guarantees that the player will win the game.

Strategies

- Playing according to a PI strategy for One:

I	$\sigma(\emptyset)$	$\sigma(x_0)$	$\sigma(x_0, x_1)$	$\cdots$
II	x_0	x_1	x_2	$\cdots$

- Playing according to a PI strategy for Two:

I	A_0	A_1	A_2	$\cdots$
II	$\tau(A_0)$	$\tau(A_0, A_1)$	$\tau(A_0, A_1, A_2)$	$\cdots$

- Playing according to a Markov strategy for Two:

I	A_0	A_1	A_2	$\cdots$
II	$\tau(A_0, 0)$	$\tau(A_1, 1)$	$\tau(A_2, 2)$	$\cdots$

- Playing according to a PD strategy for One:

I	$\sigma(0)$	$\sigma(1)$	$\sigma(2)$	$\cdots$
II	x_0	x_1	x_2	$\cdots$

A Strength Hierarchy

For the selection game $G_1(\mathcal{A}, \mathcal{B})$:

$$\begin{array}{rcl} \text{Two has a winning Markov Strategy} & & \\ \Downarrow & & \\ \text{Two has a winning PI strategy} & & \\ \Downarrow & & \\ \text{One has no winning PI strategy} & & \\ \Downarrow & & \\ \text{One has no winning PD strategy} & \iff & S_1(\mathcal{A}, \mathcal{B}) \end{array}$$

Game Equivalence

Definition

Define $G_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_1(\mathcal{B}, \mathcal{D})$ as the conjunction of the following implications.

1. Two has Mark in $G_1(\mathcal{A}, \mathcal{C}) \implies$ Two has a Mark in $G_1(\mathcal{B}, \mathcal{D})$
2. Two has PI strat in $G_1(\mathcal{A}, \mathcal{C}) \implies$ Two has a PI strat in $G_1(\mathcal{B}, \mathcal{D})$
3. One has no PI strat in $G_1(\mathcal{A}, \mathcal{C}) \implies$ One has no PI strat in $G_1(\mathcal{B}, \mathcal{D})$
4. One has no PD strat in $G_1(\mathcal{A}, \mathcal{C}) \implies$ One has no PD strat in $G_1(\mathcal{B}, \mathcal{D})$

- This relation is transitive.
- if $G_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_1(\mathcal{B}, \mathcal{D})$ and $G_1(\mathcal{B}, \mathcal{D}) \leq_{\text{II}} G_1(\mathcal{A}, \mathcal{C})$, we say the two games are **equivalent** and write $G_1(\mathcal{A}, \mathcal{C}) \equiv G_1(\mathcal{B}, \mathcal{D})$.
- There is a G_{fin} version of all of this.

Scheeper's Monotoncity Law

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be collections.

Monotonicity

- If $\mathcal{B} \subseteq \mathcal{A}$, then $G_\square(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_\square(\mathcal{B}, \mathcal{C})$.
- If $\mathcal{D} \subseteq \mathcal{C}$, then $G_\square(\mathcal{A}, \mathcal{D}) \leq_{\text{II}} G_\square(\mathcal{A}, \mathcal{C})$.

As an application of this law, $G_1(\Omega_{X,x}, \Omega_{X,x}) \leq_{\text{II}} G_1(\mathcal{D}_X, \Omega_{X,x})$ for any space X and point $x \in X$.

Bijective Translation

Caruvana, Holshouser 2020

Assume that $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are collections and that $\bigcup \mathcal{C} \subseteq \bigcup \mathcal{A}$ and $\bigcup \mathcal{D} \subseteq \bigcup \mathcal{B}$. Suppose that there is a bijection $\beta : \bigcup \mathcal{A} \to \bigcup \mathcal{B}$ so that

- $A \in \mathcal{A}$ if and only if $\beta[A] \in \mathcal{B}$, and
- $C \in \mathcal{C}$ if and only if $\beta[C] \in \mathcal{D}$.

Then $G_\square(\mathcal{A}, \mathcal{C}) \equiv G_\square(\mathcal{B}, \mathcal{D})$.

Using Bijective Translation

Let $\mathbb{F}(X)$ denote the Fell topology on the hyperspace of closed subsets of X and let $\mathcal{K}_F(X)$ be the collection of k_F-covers of X (as defined by Li).

Caruvana, Holshouser 2020

$G_\square(\mathcal{K}_F(X), \mathcal{K}_F(X)) \equiv G_\square(\mathcal{D}_{\mathbb{F}(X)}, D_{\mathbb{F}(X)})$.

We can prove this by defining $\beta : \mathcal{T}_X \to \mathbb{F}(X)$ as $\beta(U) = X \setminus U$ and then confirming that $\mathscr{U} \in \mathcal{K}_F(X)$ if and only if $\beta[\mathscr{U}] \in \mathcal{D}_{\mathbb{F}(X)}$.

Turn Based Translation

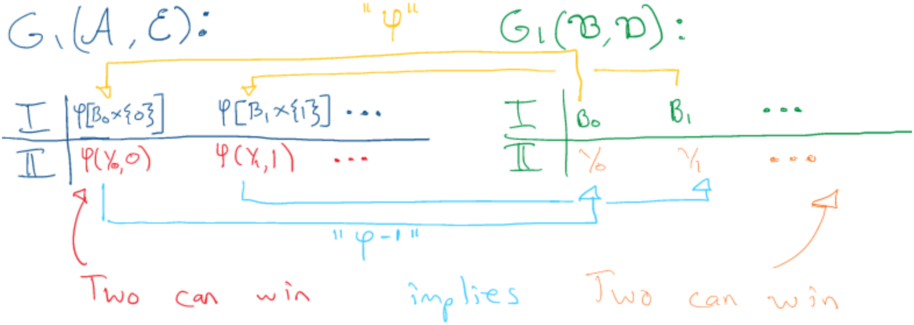
Caruvana, Holshouser 2020

Assume that $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are collections and that $\bigcup \mathcal{C} \subseteq \bigcup \mathcal{A}$ and $\bigcup \mathcal{D} \subseteq \bigcup \mathcal{B}$. Suppose that there is a function $\varphi : \bigcup \mathcal{B} \times \omega \to \bigcup \mathcal{A}$ so that

- If $B \in \mathcal{B}$ and $n \in \omega$, then $\varphi[B \times \{n\}] \in \mathcal{A}$, and
- If $y_n \in B_n$ and $\{\varphi(y_n, n) : n \in \omega\} \in \mathcal{C}$, then $\{y_n : n \in \omega\} \in \mathcal{D}$.

Then $G_1(\mathcal{A}, \mathcal{C}) \leq_{\text{II}} G_1(\mathcal{B}, \mathcal{D})$.

Turn Based Translation



① $B \in \mathcal{B}$ and $n \in \omega$ $\Rightarrow$ $\varphi[B \times \{n\}] \in \mathcal{A}$

② $Y_n \in B_n$ and $\{\varphi(Y_n, n) : n \in \omega\} \in \mathcal{E}$ $\Rightarrow$ $\{Y_n : n \in \omega\} \in \mathcal{D}$

Using Turn Based Translation

Let X be a topological space, $\mathcal{K}(X)$ be the k-covers of X, and $C_K(X)$ be the continuous functions from X to $\mathbb{R}$ with the compact-open topology.

Caruvana, Holshouser 2020

$G_1(\mathcal{K}(X), \mathcal{K}(X)) \leq_{\text{II}} G_1(\Omega_{C_k(X),\mathbf{0}}, \Omega_{C_k(X),\mathbf{0}})$.

We can prove this by defining $\varphi : C_K(X) \times \omega \to \mathcal{T}_X$ as $\varphi(f, n) = f^{-1}(-2^{-n}, 2^{-n})$ and then confirming that

- If $\mathbf{0} \in \overline{A}$, then $\{f^{-1}(-2^{-n}, 2^{-n}) : f \in A\}$ is a k-cover, and
- If $\{f_n^{-1}(-2^{-n}, 2^{-n}) : n \in \omega\}$ is a k-cover, then $\mathbf{0} \in \overline{\{f_n : n \in \omega\}}$.

General Translation

Theorem (Caruvana and Holshouser 2019)

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be collections. Suppose there are functions

- $\overleftarrow{T}_{\mathrm{I},n} : \mathcal{B} \to \mathcal{A}$ and
- $\overrightarrow{T}_{\mathrm{II},n} : \bigcup \mathcal{A} \times \mathcal{B} \to \bigcup \mathcal{B}$

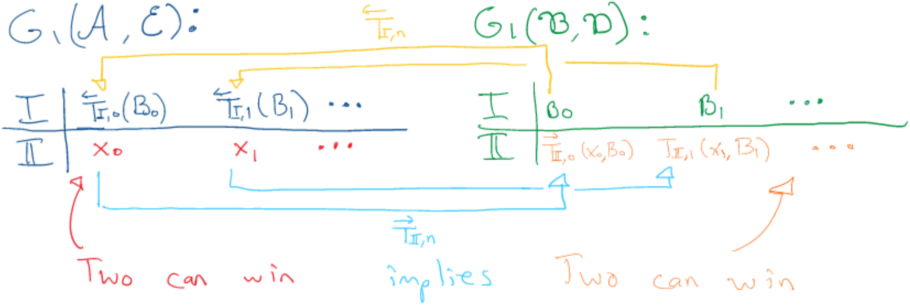
so that

1. if $x \in \overleftarrow{T}_{\mathrm{I},n}(B)$, then $\overrightarrow{T}_{\mathrm{II},n}(x, B) \in B$ and
2. if $x_n \in \overleftarrow{T}_{\mathrm{I},n}(B_n)$ for all n, then

$$\{x_n : n \in \omega\} \in \mathcal{C} \implies \left\{\overrightarrow{T}_{\mathrm{II},n}(x_n, B_n) : n \in \omega\right\} \in \mathcal{D}$$

Then $G_1(\mathcal{A}, \mathcal{C}) \leq_{\mathrm{II}} G_1(\mathcal{B}, \mathcal{D})$.

General Translation



① $x \in \overleftarrow{T}_{I,n}(B) \Rightarrow \overrightarrow{T}_{II,n}(x, B) \in B$

② $x_n \in \overleftarrow{T}_{I,n}(B_n)$ and $\{x_n : n \in \omega\} \in \mathcal{E} \Rightarrow \{\overrightarrow{T}_{II,n}(x_n, B_n) : n \in \omega\} \in \mathcal{D}$

Using General Translation

Let X be a $T_{3.5}$ topological space, $\mathcal{K}(X)$ be the k-covers of X, and $C_K(X)$ be the continuous functions from X to $\mathbb{R}$ with the compact-open topology.

Theorem (Caruvana and Holshouser 2019)

$G_1(\mathcal{D}_{C_k(X)}, \Omega_{C_K(X),\mathbf{0}}) \leq_{\mathrm{II}} G_1(\mathcal{K}(X), \mathcal{K}(X))$

Define $\overleftarrow{T}_{\mathrm{I},n}(\mathscr{U}) = \{f : (\exists U \in \mathscr{U})[f[X \setminus U] = \{1\}\}$ and set $\overrightarrow{T}_{\mathrm{II},n}(f, \mathscr{U})$ to be a choice of $U \in \mathscr{U}$ so that $f[X \setminus U] = \{1\}$ if possible, and X otherwise. Confirm that

- $\overleftarrow{T}_{\mathrm{I},n}$ and $\overrightarrow{T}_{\mathrm{II},n}$ are well-defined,
- (The first condition immediately falls out), and
- If $\mathscr{U}_n$ are k-covers, f_n and $U_n \in \mathscr{U}_n$ are so that $f_n[X \setminus U_n] = \{1\}$, and $\mathbf{0} \in \overline{\{f_n : n \in \omega\}}$, then $\{U_n : n \in \omega\}$ is a k-cover.

A Game Reduction That Does Not Use Translation

Let X be a topological space, $\mathcal{O}$ be the open covers of X, and Ω be the ω-covers of X. Then

Theorem

$G_1(\Omega, \Omega) \leq_{\mathrm{II}} G_1(\mathcal{O}, \mathcal{O})$.

- Player II has a winning (Markov) strategy in $G_1(\Omega, \Omega)$ if and only if they have a winning (Markov) strategy in $G_1(\mathcal{O}, \mathcal{O})$.
- Using the result of Pawlikowski, Player I has a winning strat in $G_1(\mathcal{O}, \mathcal{O})$ if and only if they have winning PD strat.
- Using the generalization of Pawlikowski, Player I has a winning strat in $G_1(\Omega, \Omega)$ if and only if they have winning PD strat.
- Using the bijection between ω and ω^2, $S_1(\Omega, \Omega)$ is equivalent $S_1(\mathcal{O}, \mathcal{O})$.

Brittle Game Reductions

- The game reductions that came from translations all generalize to different ideals. In particular, they work both in reference to compact sets and to finite sets.
- $G_1(\Omega, \Omega) \leq_{\text{II}} G_1(\mathcal{O}, \mathcal{O})$, but $G_1(\mathcal{K}, \mathcal{K}) \not\leq_{\text{II}} G_1(\mathcal{O}, \mathcal{O})$.
- The game reductions that come from translation can either be directly applied, or slightly modified, to establish a reduction between the finite selection games, groupable versions of the games, longer games, and localized versions of the games.
- $G_{\text{fin}}(\Omega, \Omega) \leq_{\text{II}} G_{\text{fin}}(\mathcal{O}, \mathcal{O})$ and $G_{\text{fin}}(\mathcal{K}, \mathcal{K}) \leq_{\text{II}} G_{\text{fin}}(\mathcal{O}, \mathcal{O})$.

Thanks!

Thanks for Listening